

# Strong convergence rates and temporal regularity for Cox-Ingersoll-Ross processes and Bessel processes with accessible boundaries

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## Abstract

Cox-Ingersoll-Ross (CIR) processes are widely used in financial modeling such as in the Heston model for the approximative pricing of financial derivatives. Moreover, CIR processes are mathematically interesting due to the irregular square root function in the diffusion coefficient. In the literature, positive strong convergence rates for numerical approximations of CIR processes have been established in the case of an inaccessible boundary point. Since calibrations of the Heston model frequently result in parameters such that the boundary is accessible, we focus on this interesting case. Our main result shows for every  $p \in (0, \infty)$  that the drift-implicit square-root Euler approximations proposed in Alfonsi (2005) converge in the strong  $L^p$ -distance with a positive rate for half of the parameter regime in which the boundary point is accessible. A key step in our proof is temporal regularity of Bessel processes. More precisely, we prove for every  $p \in (0, \infty)$  that Bessel processes are temporally  $1/2$ -Hölder continuous in  $L^p$ .

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## 1 Introduction

Cox-Ingersoll-Ross (CIR) processes have been proposed in Cox, Ingersoll & Ross [7] as a model for short-term interest rates. Since then, CIR processes are widely used in financial modeling. For instance, CIR processes appear as the instantaneous squared volatility in the Heston model [21] which is one of the most popular equity derivatives pricing models among practitioners and which is day after day being numerically approximated in the financial engineering industry. Building on explicitly known Fourier transforms of CIR processes at fixed time points, it is well known how to value plain vanilla options in the Heston model or how to calibrate the Heston model with European calls and puts (see, however, Kahl & Jäckel [25] for numerical issues). In contrast, complicated path-dependent financial derivatives within the Heston model are prized by using time-discrete approximations of the Heston model and Monte Carlo methods. In addition, positive strong convergence rates of the time-discrete approximations are important for applying efficient multilevel Monte Carlo methods (see Giles [15], Kebaier [26], Heinrich [19, 20]). In view of this, the central goal of this article is to establish a positive strong convergence rate for time-discrete approximations of CIR processes.

For a formal introduction of CIR processes, let  $x, \beta \in (0, \infty)$ ,  $\delta \in [0, \infty)$ ,  $\gamma \in \mathbb{R}$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, \infty)})$  be a stochastic basis (see Section 1.1 for this and further notation), let  $W: [0, \infty) \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, \infty)}$ -Wiener process and let  $X: [0, \infty) \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying the stochastic differential equation (SDE)

$$dX_t = (\delta - \gamma X_t) dt + \beta \sqrt{X_t} dW_t, \quad t \in [0, \infty), \quad X_0 = x. \quad (1)$$

If  $\delta, \gamma \in (0, \infty)$ , then the processes  $X$  is the CIR process with long-time mean  $\delta/\gamma$ , speed of adaptation  $\gamma$  and volatility  $\beta$ . Moreover if  $\gamma = 0$  and  $\beta = 2$ , then  $X$  is the squared Bessel process and  $\sqrt{X}$  is the *Bessel process of “dimension”*  $\delta$ ; see, e.g., Göing-Jaeschke & Yor [16] for a review on Bessel processes. We note that according to Feller’s boundary classification (see, e.g., Theorem V.51.2 in [33]), the boundary point zero is inaccessible (that is,  $\mathbb{P}[\forall t \in (0, \infty): X_t > 0] = 1$ ) if and only if  $2\delta \geq \beta^2$ . Alfonsi [1] proposed in the case  $4\delta > \beta^2$  the numerical approximations  $Y^h: [0, \infty) \times \Omega \rightarrow [0, \infty)$ ,  $h \in (0, \infty) \cap (0, 2/\gamma^-)$ , satisfying for all  $h \in (0, 2/\gamma^-)$ ,  $n \in \mathbb{N}_0$ ,  $t \in (nh, (n+1)h]$  that  $Y_0^h = X_0$  and

$$Y_t^h = \frac{(n+1)h-t}{h} Y_{nh}^h + \frac{t-nh}{h} \left[ \frac{(Y_{nh}^h)^{1/2} + \frac{\beta}{2}(W_{(n+1)h} - W_{nh}) + \sqrt{[(Y_{nh}^h)^{1/2} + \frac{\beta}{2}(W_{(n+1)h} - W_{nh})]^2 + (2+\gamma h)(\delta - \frac{\beta^2}{4})h}}{(2+\gamma h)} \right]^2. \quad (2)$$

Following Dereich, Neuenkirch & Szpruch [11], we refer to the numerical approximations (2) as *linearly interpolated drift-implicit square-root Euler approximations*.

In the literature, the goal of establishing positive strong convergence rates for CIR processes has been achieved in the case of an inaccessible boundary point. The first positive rates were established in Theorem 2.2 in Berkaoui, Bossy & Diop [5] which proves for every  $p \in [1, \infty)$  with  $\frac{2\delta}{\beta^2} > 1 + \sqrt{8} \max(\frac{1}{\beta} \sqrt{\gamma^+ (16p-1)}, 16p-2)$  the uniform  $L^p$ -rate  $1/2$  in the case of a symmetrized Euler scheme. In addition, Theorem 1.1 in Dereich, Neuenkirch & Szpruch [11] implies that if the boundary point zero is inaccessible, then the linearly interpolated drift-implicit square-root Euler approximations (2) converge for every  $p \in [1, \frac{2\delta}{\beta^2})$  with uniform  $L^p$ -rate  $1/2$ -. Theorem 2 in Alfonsi [2] and Proposition 3.1 in Neuenkirch & Szpruch [29] even imply for every  $p \in [1, \frac{4\delta}{3\beta^2})$  the uniform  $L^p$ -rate 1 in the regime  $\delta > \beta^2$ . These results exploit that the process  $Z: [0, \infty) \times \Omega \rightarrow [0, \infty)$  defined through  $Z_t := \sqrt{X_t}$ ,  $t \in [0, \infty)$ , satisfies in the case  $2\delta \geq \beta^2$  the SDE with additive noise

$$dZ_t = \left[ \frac{4\delta - \beta^2}{8Z_t} - \frac{\gamma}{2} Z_t \right] dt + \frac{\beta}{2} dW_t, \quad t \in [0, \infty), \quad (3)$$

which follows from Itô’s lemma applied to the  $C^2((0, \infty), \mathbb{R})$ -function  $(0, \infty) \ni x \mapsto \sqrt{x} \in \mathbb{R}$ . The reason for considering the SDE (3) is that if the boundary point zero is inaccessible, then the SDE (3) has a globally one-sided Lipschitz continuous drift coefficient and a globally Lipschitz continuous diffusion coefficient. To the best of our knowledge, the above goal remained an open problem when the boundary point zero is accessible. For results on strong convergence without rate which apply to CIR processes with accessible boundary, see, e.g., Deelstra & Delbaen [10], Alfonsi [1], Higham & Mao [22], Lord, Koekkoek & Dijk [27], Gyöngy & Rásonyi [17], Halidias [18]. For results on pathwise convergence which apply to CIR processes with accessible boundary, see, e.g., Milstein & Schoenmakers [28]. To establish strong convergence rates in the case  $2\delta < \beta^2$  is of particular difficulty as the coefficients of the SDE (1) are not even *locally* Lipschitz continuous on the state space in that case.

In real-world applications of the Heston model, calibrations frequently result in parameters such that  $2\delta < \beta^2$  so that zero is an accessible boundary point of the volatility process; see, e.g., Table III in [3], Table 1 in [12] and Table 1 in [13] for calibration results in the literature. So it is important for practitioners to know which numerical approximations converge with a positive rate even in the case of an accessible boundary point. To the best of our knowledge, our main theorem, Theorem 1.1, is the first result which establishes a positive rate of strong convergence of numerical approximations of Cox-Ingersoll-Ross processes which hit the boundary point 0 with positive probability.

**Theorem 1.1.** *Assume the above setting, assume  $2\delta/\beta^2 > 1/2$  and let  $T, \varepsilon \in (0, \infty)$ ,  $p \in [1, \infty)$ . Then there exists a real number  $C \in [0, \infty)$  such that for all  $h \in (0, T] \cap (0, 1/2\gamma^-)$  it holds that*

$$\left( \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - Y_t^h|^p \right] \right)^{1/p} \leq C \cdot h^{\left[ \frac{(2\delta/\beta^2) \wedge 1 - 1/2}{p} - \varepsilon \right]}. \quad (4)$$

Theorem 1.1 follows from Corollary 3.15 below. Corollary 3.15 in turn is a corollary of Theorem 3.13 which proves a positive rate of strong convergence for drift-implicit square-root Euler approximations for more general square-root diffusion processes with accessible boundaries under the assumption of certain inverse moments. In addition, Corollary 3.9 below and Theorem 2.13 below yield a strong approximation result for the Bessel process  $Z$ . We emphasize that it is not clear to us whether the strong convergence rate proved in Theorem 1.1 is sharp. It might very well be the case that the rate is not sharp (see the numerical simulation in Figure 2 in Alfonsi [1] which suggests an  $L^1$ -rate between  $1/2$  and 1 for  $2\delta/\beta^2 \in (\frac{1}{2}, 1)$ ).

We sketch the main steps in our proof of Theorem 1.1. Corollary 3.9 below shows that suitable uniform temporal Hölder regularity of the Bessel-type processes (3) together with uniform moment bounds of the numerical approximation processes (2) is actually enough to deduce a strong rate of convergence of the linearly interpolated drift-implicit square-root Euler approximations (2). So this temporal Hölder regularity of the Bessel-type processes (3) is an important step in our proof of Theorem 1.1 and is subject of the following theorem, Theorem 1.2. To the best of our knowledge, Theorem 1.2 (choose  $\beta = 2$  and  $\gamma = 0$ ) is the first result which proves for every  $q \in (0, \infty)$ ,  $\varepsilon \in (0, \frac{1}{2})$  that  $Z \in L^q(\Omega; C^{1/2-\varepsilon}([0, T], [0, \infty)))$  holds for the Bessel process  $Z$  with “dimension”  $\delta \in [0, 2)$ ; see, e.g., Lemma 3.2 in Dereich, Neuenkirch & Szpruch [11] for the case  $\frac{4\delta}{\beta^2} \in (2, \infty)$ .

**Theorem 1.2.** *Assume the above setting. Then for all  $T, \varepsilon, p \in (0, \infty)$  it holds that*

$$\sup_{\substack{s, t \in [0, T] \\ s \neq t}} \left\| \frac{Z_t - Z_s}{|t-s|^{1/2}} \right\|_{L^p(\Omega; \mathbb{R})} + \left\| \sup_{\substack{s, t \in [0, T] \\ s \neq t}} \frac{|Z_t - Z_s|}{|t-s|^{1/2-\varepsilon}} \right\|_{L^p(\Omega; \mathbb{R})} < \infty. \quad (5)$$

Theorem 1.2 follows from Corollary 2.14 below. Corollary 2.14 in turn is a corollary of Theorem 2.13 which proves temporal  $1/2$ -Hölder continuity for more general diffusion processes with additive noise (see also Lemma 3.2 below).

## 1.1 Notation

Throughout this article we use the following notation. For  $d \in \mathbb{N} := \{1, 2, \dots\}$  and  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ , we denote by  $\|v\|_{\mathbb{R}^d} := [|v_1|^2 + \dots + |v_d|^2]^{1/2}$  the Euclidean norm of  $v$ . For two sets  $A$  and  $B$  we denote by  $\mathcal{M}(A, B)$  the set of all mappings from  $A$  to  $B$ . For two measurable spaces  $(A, \mathcal{A})$  and  $(B, \mathcal{B})$  we denote by  $\mathcal{L}^0(A; B)$  the set of all  $\mathcal{A}/\mathcal{B}$ -measurable mappings from  $A$  to  $B$ . If  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space, if  $I \subseteq \mathbb{R}$  is a closed and non-empty interval and if  $(\mathcal{F}_t)_{t \in I}$  is a normal filtration on  $(\Omega, \mathcal{F}, \mathbb{P})$ , then we call the quadrupel  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$  a *stochastic basis* (cf., e.g., Appendix E in Prévôt & Röckner [30]). For an interval  $O \subseteq \mathbb{R}$  with  $\#(O) = \infty$  and two functions  $\mu: O \rightarrow \mathbb{R}$  and  $\sigma: O \rightarrow \mathbb{R}$ , we define the linear operator  $\mathcal{G}_{\mu, \sigma}: C^2(O, \mathbb{R}) \rightarrow \mathcal{M}(O, \mathbb{R})$  by

$$(\mathcal{G}_{\mu, \sigma}\phi)(x) := \phi'(x)\mu(x) + \frac{1}{2}\phi''(x)(\sigma(x))^2 \quad (6)$$

for all  $x \in O$ ,  $\phi \in C^2(O, \mathbb{R})$ . Throughout this article we also often calculate and formulate expressions in the extended real numbers  $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ . In particular, we frequently use the conventions  $\sup(\emptyset) = -\infty$ ,  $\frac{0}{0} = 0 \cdot \infty = 0$ ,  $0^0 = 1$ ,  $\frac{a}{0} = \infty$ ,  $\frac{-a}{0} = -\infty$ ,  $\frac{b}{\infty} = 0$ ,  $0^a = 0$ ,  $0^{-a} = \frac{1}{0^a} = \infty$  for all  $a \in (0, \infty)$ ,  $b \in \mathbb{R}$ . Furthermore, we define  $x \wedge y := \min(x, y)$  and  $x \vee y := \max(x, y)$  for all  $x, y \in \mathbb{R}$ . Finally, if  $T \in (0, \infty)$ ,  $s \in [0, T]$ ,  $n \in \mathbb{N}$ ,  $\theta = (t_0, \dots, t_n) \in [0, T]^{n+1}$  satisfy  $0 = t_0 < t_1 < \dots < t_n = T$ , then we define  $|\theta| := \max_{k \in \{1, \dots, n\}} |t_k - t_{k-1}|$ ,  $\lfloor s \rfloor_\theta := \sup(\{t_0, t_1, \dots, t_n\} \cap [0, s])$  and  $\lceil s \rceil_\theta := \inf(\{t_0, t_1, \dots, t_n\} \cap [s, T])$ . For two normed vector spaces  $(V_1, \|\cdot\|_{V_1})$  and  $(V_2, \|\cdot\|_{V_2})$ , a real number  $\theta \in [0, \infty)$  and a function  $f: V_1 \rightarrow V_2$  from  $V_1$  to  $V_2$ , we define  $\|f\|_{\mathcal{C}^\theta(V_1, V_2)} := \sup_{v, w \in V_1} \frac{\|f(v) - f(w)\|_{V_2}}{\|v - w\|_{V_1}^\theta}$ . For a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , a normed vector space  $(V, \|\cdot\|_V)$ , an  $\mathcal{F}/\mathcal{B}(V)$ -measurable mapping  $X: \Omega \rightarrow V$  and a real number  $p \in (0, \infty)$ , we define  $\|X\|_{L^0(\Omega; V)} := 1$  and  $\|X\|_{L^p(\Omega; V)} := (\mathbb{E}[\|X\|_V^p])^{1/p} \in [0, \infty]$ .

## 2 Temporal Hölder regularity

The main results of this section, Proposition 2.9 and Theorem 2.13 below, establish temporal Hölder regularity for Bessel-type processes. A central step in the proof of Proposition 2.9 is Lemma 2.3 below which proves exponential inverse moments for Bessel-type processes.

### 2.1 Setting

Throughout Section 2 we will frequently use the following setting. Let  $T \in [0, \infty)$ ,  $\mu \in \mathcal{L}^0([0, \infty); \mathbb{R})$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying  $\int_0^T |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s. and  $X_t = X_0 + \int_0^t \mu(X_s) ds + W_t$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ .

**Remark 2.1.** Observe that if  $\sigma \in (0, \infty)$  and if  $Y: [0, T] \times \Omega \rightarrow \mathbb{R}$  is an adapted stochastic process with continuous sample paths satisfying  $Y_t = Y_0 + \int_0^t \mu(Y_s) ds + \sigma W_t$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , then the process  $\tilde{X}: [0, T] \times \Omega \rightarrow \mathbb{R}$  given by  $\tilde{X}_t = \frac{1}{\sigma} Y_t$  for all  $t \in [0, T]$  is an adapted stochastic process with continuous sample paths which satisfies  $\tilde{X}_t = \tilde{X}_0 + \int_0^t (\frac{1}{\sigma} \mu)(\tilde{X}_s) ds + W_t$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . This illustrates that w.l.o.g. one may assume that  $X$  is a solution of an SDE with a diffusion coefficient which is identically equal to the constant 1.

### 2.2 Exponential inverse moments for Bessel-type processes

**Lemma 2.2.** Assume the setting in Section 2.1, let  $c, \beta \in [0, \infty)$  and assume  $x\mu(x) \leq c(1 + x^2)$  for all  $x \in (0, \infty)$ . Then it holds for all  $t \in [0, T]$  that

$$\mathbb{E} \left[ \exp \left( \frac{1 + (X_t)^2}{e^{2t(c+1)+t\beta}} + \int_0^t \frac{\beta(1 + (X_s)^2)}{e^{2s(c+1)+s\beta}} ds \right) \right] \leq \mathbb{E} \left[ \exp(1 + (X_0)^2) \right]. \quad (7)$$

*Proof of Lemma 2.2.* First of all, we define functions  $\sigma: [0, \infty) \rightarrow \mathbb{R}$ ,  $U, \bar{U}: \mathbb{R} \rightarrow \mathbb{R}$  by  $\sigma(x) = 1$  for all  $x \in [0, \infty)$  and by  $U(x) := 1 + x^2$  and  $\bar{U}(x) := \beta(1 + x^2)$  for all  $x \in \mathbb{R}$ . Then observe that for all  $x \in [0, \infty)$  it holds that  $|\nabla U(x)|^2 = 4x^2$  and

$$(\mathcal{G}_{\mu, \sigma} U)(x) = 2x\mu(x) + 1 \leq 2c(1 + x^2) + 1. \quad (8)$$

This implies that for all  $x \in [0, \infty)$  it holds that

$$\begin{aligned} (\mathcal{G}_{\mu, \sigma} U)(x) + \frac{1}{2} |\nabla U(x)|^2 + \bar{U}(x) &\leq 2c(1 + x^2) + 1 + 2x^2 + \beta(1 + x^2) \\ &\leq (2(c+1) + \beta)(1 + x^2). \end{aligned} \quad (9)$$

Corollary 2.4 in Cox et. al [8] together with (9) shows that for all  $t \in [0, T]$  it holds that

$$\mathbb{E} \left[ \exp \left( \frac{1 + (X_t)^2}{e^{2t(c+1)+t\beta}} + \int_0^t \frac{\beta(1 + (X_s)^2)}{e^{2s(c+1)+s\beta}} ds \right) \right] \leq \mathbb{E} \left[ \exp(1 + (X_0)^2) \right]. \quad (10)$$

This finishes the proof of Lemma 2.2.  $\square$

Theorem 3.1 in Hurd & Kuznetsov [23], in particular, implies in the setting of the introduction for the Bessel-type process  $Z$  that if  $2\delta > \beta^2$  and if  $\mathbb{E}[(Z_0)^{(1-\frac{2\delta}{\beta^2})}] < \infty$ , then  $\mathbb{E}[\frac{\beta^2}{8}(\frac{2\delta}{\beta^2} - 1)^2 \int_0^T (Z_s)^{-2} ds] < \infty$ . Inequality (12) in Lemma 2.3 below together with Lemma 2.2 above complements this result with exponential inverse moments in the case  $\frac{\beta^2}{2} < 2\delta \leq \beta^2$ . In particular, we reveal in Lemma 2.3 a suitable exponentially growing *Lyapunov-type function* for Bessel- and Cox-Ingersoll-Ross-type processes respectively (see (11) and (18) below).

**Lemma 2.3.** *Assume the setting in Section 2.1, let  $\alpha \in (0, \infty)$ ,  $p \in (1, (1+2\alpha) \wedge 2)$ ,  $\tilde{c} \in [0, \infty)$ ,  $c \in [p, \infty)$  and assume  $\alpha - \tilde{c}x^c \leq x\mu(x)$  for all  $x \in (0, \infty)$ . Then it holds for all  $t \in [0, T]$ ,  $q, \rho \in (0, \infty)$ ,  $\tilde{\rho} \in (0, \rho)$ ,  $r \in (1, \infty)$  that*

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( \tilde{\rho}(2-p) \left( \alpha + \frac{1-p}{2} \right) \int_0^t (X_s)^{-p} ds - \tilde{c}\rho(2-p) \int_0^t (X_s)^{(c-p)} ds - \rho(X_t)^{(2-p)} \right) \right] \\ & \leq \exp \left( \frac{t\rho^2}{2} \left[ \frac{\rho^2}{(\rho-\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} \right) \mathbb{E} \left[ e^{-\rho(X_0)^{(2-p)}} \right] < \infty, \end{aligned} \quad (11)$$

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( \int_0^t \tilde{\rho}(2-p) \left( \alpha + \frac{1-p}{2} \right) (X_s)^{-p} ds \right) \right] \leq \exp \left( \frac{tr\rho^2}{2} \left[ \frac{r\rho^2}{(\rho-\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} \right) \\ & \cdot \left\| e^{-\rho(X_0)^{(2-p)}} \right\|_{L^r(\Omega; \mathbb{R})} \left\| \exp \left( \rho(X_t)^{(2-p)} + \int_0^t \tilde{c}\rho(2-p) (X_s)^{(c-p)} ds \right) \right\|_{L^{r/(r-1)}(\Omega; \mathbb{R})} \quad \text{and} \end{aligned} \quad (12)$$

$$\begin{aligned} & \left\| \int_0^t (X_s)^{-p} ds \right\|_{L^q(\Omega; \mathbb{R})} \leq \frac{2^{1 \vee (1/q)}}{\tilde{\rho}(2-p)(2\alpha+1-p)} \left[ \rho \left\| (X_t)^{(2-p)} + \int_0^t \tilde{c}(2-p) (X_s)^{(c-p)} ds \right\|_{L^q(\Omega; \mathbb{R})} \right. \\ & \quad \left. + \exp \left( \frac{tq\rho^2}{2} \left[ \frac{q\rho^2}{(\rho-\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} - 1 \right) \left\| e^{-\rho(X_0)^{(2-p)}} \right\|_{L^q(\Omega; \mathbb{R})} \right]. \end{aligned} \quad (13)$$

*Proof of Lemma 2.3.* Throughout this proof we fix  $\rho \in (0, \infty)$ ,  $\tilde{\rho} \in (0, \rho)$ , we define a function  $\sigma: [0, \infty) \rightarrow \mathbb{R}$  by  $\sigma(x) = 1$  for all  $x \in [0, \infty)$  and we define

$$\tilde{q} := \frac{p}{2(p-1)} \in (1, \infty), \quad \tilde{p} := \frac{1}{(1-1/\tilde{q})} \in (1, \infty), \quad \delta := \left[ \frac{2\tilde{q}(\rho-\tilde{\rho})}{(2-p)\rho^2} \left( \alpha + \frac{1-p}{2} \right) \right]^{1/\tilde{q}} \in (0, \infty). \quad (14)$$

Then we observe that Young's inequality shows that for all  $\varepsilon \in (0, 1)$ ,  $x \in [0, \infty)$  it holds that

$$(\varepsilon + x)^{(2-2p)} \leq \frac{1}{\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} + \frac{\delta^{\tilde{q}} (\varepsilon + x)^{(2-2p)\tilde{q}}}{\tilde{q}} = \frac{1}{\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} + \frac{\delta^{\tilde{q}}}{\tilde{q} (\varepsilon + x)^{\tilde{p}}}. \quad (15)$$

In the next step we define functions  $U_\varepsilon, \overline{U}_\varepsilon: [0, \infty) \rightarrow \mathbb{R}$ ,  $\varepsilon \in (0, 1)$ , by  $U_\varepsilon(x) := -\rho(\varepsilon + x)^{(2-p)}$  and

$$\overline{U}_\varepsilon(x) := \tilde{\rho}(2-p) \left( \alpha + \frac{1-p}{2} \right) (\varepsilon + x)^{-p} - \tilde{c}\rho(2-p) (\varepsilon + x)^{(c-p)} - \frac{\rho^2(2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} \quad (16)$$

for all  $x \in [0, \infty)$  and all  $\varepsilon \in (0, 1)$ . Observe that for all  $\varepsilon \in (0, 1)$  it holds that  $U_\varepsilon \in C^2([0, \infty), \mathbb{R})$  and that for all  $x \in [0, \infty)$ ,  $\varepsilon \in (0, 1)$  it holds that  $|(\nabla U_\varepsilon)(x)|^2 = \rho^2(2-p)^2(\varepsilon + x)^{(2-2p)}$  and

$$(\mathcal{G}_{\mu, \sigma} U_\varepsilon)(x) = -\rho(2-p)(\varepsilon + x)^{(1-p)}\mu(x) - \rho(2-p)\frac{(1-p)}{2}(\varepsilon + x)^{-p}. \quad (17)$$

Moreover, note that definition (15) ensures that  $\frac{\rho^2(2-p)}{2\tilde{q}}\delta^{\tilde{q}} + (\tilde{\rho} - \rho) \left( \alpha + \frac{1-p}{2} \right) = 0$ . This, the fact that for all  $x \in (0, \infty)$  it holds that  $-\mu(x) \leq \tilde{c}x^{(c-1)} - \frac{\alpha}{x} \leq \tilde{c}(\varepsilon + x)^{(c-1)} - \frac{\alpha}{(\varepsilon + x)}$  and (15) imply that for all  $x \in [0, \infty)$ ,  $\varepsilon \in (0, 1)$  it holds that

$$\begin{aligned} & (\mathcal{G}_{\mu, \sigma} U_\varepsilon)(x) + \frac{1}{2} |(\nabla U_\varepsilon)(x)|^2 + \overline{U}_\varepsilon(x) \\ & = -\rho(2-p)(\varepsilon + x)^{(1-p)}\mu(x) - \rho(2-p)\frac{(1-p)}{2}(\varepsilon + x)^{-p} + \frac{\rho^2(2-p)^2}{2}(\varepsilon + x)^{(2-2p)} \\ & \quad + \tilde{\rho}(2-p) \left( \alpha + \frac{1-p}{2} \right) (\varepsilon + x)^{-p} - \tilde{c}\rho(2-p) (\varepsilon + x)^{(c-p)} - \frac{\rho^2(2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} \\ & \leq -\alpha\rho(2-p)(\varepsilon + x)^{-p} + \tilde{c}\rho(2-p) (\varepsilon + x)^{(c-p)} - \rho(2-p)\frac{(1-p)}{2}(\varepsilon + x)^{-p} \\ & \quad + \frac{\rho^2(2-p)^2}{2}(\varepsilon + x)^{(2-2p)} + \tilde{\rho}(2-p) \left( \alpha + \frac{1-p}{2} \right) (\varepsilon + x)^{-p} - \tilde{c}\rho(2-p) (\varepsilon + x)^{(c-p)} - \frac{\rho^2(2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} \\ & \leq (2-p)(\tilde{\rho} - \rho) \left( \alpha + \frac{1-p}{2} \right) (\varepsilon + x)^{-p} + \frac{\rho^2(2-p)^2}{2} \left[ \frac{1}{\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} + \frac{1}{\tilde{q}} \delta^{\tilde{q}} (\varepsilon + x)^{-p} \right] - \frac{\rho^2(2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} \\ & = \left( (2-p)(\tilde{\rho} - \rho) \left( \alpha + \frac{1-p}{2} \right) + \frac{\rho^2(2-p)^2}{2} \frac{\delta^{\tilde{q}}}{\tilde{q}} \right) (\varepsilon + x)^{-p} = 0. \end{aligned} \quad (18)$$

Corollary 2.4 in Cox et. al [8] and (18) show that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( -\rho (\varepsilon + X_t)^{(2-p)} + \int_0^t \frac{\tilde{\rho} (2-p) (\alpha + \frac{1-p}{2})}{(\varepsilon + X_s)^p} ds - \frac{\tilde{c} \rho (2-p)}{(\varepsilon + X_s)^{(p-c)}} - \frac{\rho^2 (2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} ds \right) \right] \\ & \leq \mathbb{E} \left[ \exp \left( -\rho (\varepsilon + X_0)^{(2-p)} \right) \right]. \end{aligned} \quad (19)$$

In the next step we observe that the identities  $\frac{1}{2\tilde{q}} = \frac{(p-1)}{p}$ ,  $\tilde{q} - 1 = \frac{(2-p)}{(2p-2)}$ ,  $\tilde{p} = \frac{p}{(2-p)}$  and  $\frac{\tilde{p}}{\tilde{q}} = \frac{(2p-2)}{(2-p)}$  and the estimate  $\frac{(2-p)(p-1)}{p} \leq \frac{1}{2}$  show that

$$\begin{aligned} \frac{t\rho^2 (2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} &= \frac{t\rho^2 (2-p)^2}{2\tilde{p}} \left[ \frac{\rho^2 (2-p)}{2\tilde{q} (\rho-\tilde{\rho}) (\alpha + \frac{1-p}{2})} \right]^{\frac{(2p-2)}{(2-p)}} \\ &= \frac{t\rho^2 (2-p)^3}{2p} \left[ \frac{\rho^2 (2-p)(p-1)}{(\rho-\tilde{\rho})p(\alpha + \frac{1-p}{2})} \right]^{\frac{(2p-2)}{(2-p)}} \leq \frac{t\rho^2}{2} \left[ \frac{\rho^2}{(\rho-\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}}. \end{aligned} \quad (20)$$

In the next step we note that Fatou's lemma, the dominated convergence theorem, the observation that for every  $\omega \in \Omega$  it holds that the function  $[0, T] \ni s \rightarrow (X_s(\omega))^{(c-p)} \in [0, \infty)$  is bounded from above and (19) and (20) imply that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( -\rho (X_t)^{(2-p)} + \int_0^t \tilde{\rho} (2-p) (\alpha + \frac{1-p}{2}) (X_s)^{-p} ds - \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right) \right] \\ & \leq \mathbb{E} \left[ \exp \left( -\rho (X_t)^{(2-p)} + \liminf_{(0,1) \ni \varepsilon \rightarrow 0} \left[ \int_0^t \frac{\tilde{\rho} (2-p) (\alpha + \frac{1-p}{2})}{(\varepsilon + X_s)^p} ds - \int_0^t \frac{\tilde{c} \rho (2-p)}{(\varepsilon + X_s)^{(p-c)}} ds \right] \right) \right] \\ & \leq \liminf_{(0,1) \ni \varepsilon \rightarrow 0} \mathbb{E} \left[ \exp \left( -\rho (\varepsilon + X_t)^{(2-p)} + (2-p) \int_0^t \frac{\tilde{\rho} (\alpha + \frac{1-p}{2})}{(\varepsilon + X_s)^p} ds - \tilde{c} \rho (\varepsilon + X_s)^{(c-p)} ds \right) \right] \\ & \leq \exp \left( \frac{t\rho^2 (2-p)^2}{2\tilde{p}} \left[ \frac{1}{\delta} \right]^{\tilde{p}} \right) \liminf_{(0,1) \ni \varepsilon \rightarrow 0} \mathbb{E} \left[ \exp \left( -\rho (\varepsilon + X_0)^{(2-p)} \right) \right] \\ & \leq \exp \left( \frac{t\rho^2}{2} \left[ \frac{\rho^2}{(\rho-\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} \right) \mathbb{E} \left[ \exp \left( -\rho (X_0)^{(2-p)} \right) \right]. \end{aligned} \quad (21)$$

This proves (11). In the next step observe that Hölder's inequality and (11) imply that for all  $t \in [0, T]$ ,  $r \in (1, \infty)$  it holds that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( \int_0^t \tilde{\rho} (2-p) (\alpha + \frac{1-p}{2}) (X_s)^{-p} ds \right) \right] \\ &= \mathbb{E} \left[ \exp \left( \int_0^t \frac{\tilde{\rho} (2-p) (\alpha + \frac{1-p}{2})}{(X_s)^p} ds - \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds + \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right) \right] \\ & \leq \left\| \exp \left( \int_0^t \tilde{\rho} (2-p) (\alpha + \frac{1-p}{2}) (X_s)^{-p} ds - \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds - \rho (X_t)^{(2-p)} \right) \right\|_{L^r(\Omega; \mathbb{R})} \\ & \quad \cdot \left\| \exp \left( \rho (X_t)^{(2-p)} + \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right) \right\|_{L^{r/(r-1)}(\Omega; \mathbb{R})} \\ & \leq \exp \left( \frac{t(r\rho)^2}{2r} \left[ \frac{(r\rho)^2}{(r\rho-r\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} \right) \left( \mathbb{E} \left[ e^{-r\rho (X_0)^{(2-p)}} \right] \right)^{\frac{1}{r}} \\ & \quad \cdot \left\| \exp \left( \rho (X_t)^{(2-p)} + \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right) \right\|_{L^{r/(r-1)}(\Omega; \mathbb{R})}. \end{aligned} \quad (22)$$

This shows (12). It thus remains to prove (13). For this observe that for all  $x, y \in \mathbb{R}$  it holds that

$$x \vee 0 \leq [(x-y) \vee 0] + [y \vee 0] \leq [(x-y) \vee 0] + |y| \leq \exp(x-y-1) + |y|. \quad (23)$$

This and (11) imply that for all  $t \in [0, T]$ ,  $q \in (0, \infty)$  it holds that

$$\begin{aligned} & \left\| \int_0^t \tilde{\rho} (2-p) (\alpha + \frac{1-p}{2}) (X_s)^{-p} ds \right\|_{L^q(\Omega; \mathbb{R})} \\ & \leq \left\| \exp \left( \int_0^t \frac{\tilde{\rho} (2-p) (\alpha + \frac{1-p}{2})}{(X_s)^p} ds - \rho (X_t)^{(2-p)} - \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds - 1 \right) \right. \\ & \quad \left. + \rho (X_t)^{(2-p)} + \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right\|_{L^q(\Omega; \mathbb{R})} \\ & \leq \frac{2^{0 \vee (1/q-1)}}{e} \left\| \exp \left( \int_0^t \frac{\tilde{\rho} (2-p) (\alpha + \frac{1-p}{2})}{(X_s)^p} ds - \rho (X_t)^{(2-p)} - \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right) \right\|_{L^q(\Omega; \mathbb{R})} \\ & \quad + 2^{0 \vee (1/q-1)} \left\| \rho (X_t)^{(2-p)} + \int_0^t \tilde{c} \rho (2-p) (X_s)^{(c-p)} ds \right\|_{L^q(\Omega; \mathbb{R})} \\ & \leq \frac{2^{0 \vee (1/q-1)}}{e} \exp \left( \frac{t(q\rho)^2}{2q} \left[ \frac{(q\rho)^2}{(q\rho-q\tilde{\rho})(2\alpha+1-p)} \right]^{\frac{(2p-2)}{(2-p)}} \right) \left| \mathbb{E} \left[ e^{-q\rho (X_0)^{(2-p)}} \right] \right|^{1/q} \\ & \quad + \rho 2^{0 \vee (1/q-1)} \left\| (X_t)^{(2-p)} + \int_0^t \tilde{c} (2-p) (X_s)^{(c-p)} ds \right\|_{L^q(\Omega; \mathbb{R})}. \end{aligned} \quad (24)$$

This proves (13) and thereby finishes the proof of Lemma 2.3.  $\square$

An important assumption that we use in Lemma 2.3 above and in several results below is the assumption that there exist real numbers  $\alpha, c \in (0, \infty)$ ,  $\tilde{c} \in [0, \infty)$  such that for all  $x \in (0, \infty)$  it holds that

$$\alpha - \tilde{c}x^c \leq x\mu(x). \quad (25)$$

The next remark presents a slightly modified version of this assumption (see (26)) and shows that the modified version in (26) also ensures that (25) is fulfilled.

**Remark 2.4.** Assume the setting in Section 2.1. If there exist real numbers  $\alpha \in (0, \infty)$ ,  $c \in (1, \infty)$  such that for all  $x \in (0, \infty)$  it holds that

$$\alpha - c(x^{1/c} + x^c) \leq x\mu(x), \quad (26)$$

then we obtain from Young's inequality that for all  $x, \delta \in (0, \infty)$  it holds that

$$\left[ \alpha - \delta^{1/(c^2-1)} \left[ c - \frac{1}{c} \right] \right] - \left[ \frac{1}{\delta c^2} + c \right] x^c \leq \alpha - c(x^{1/c} + x^c) \leq x\mu(x). \quad (27)$$

The next lemma (together with Lemma 3.2 below) extends Theorem 3.1 in Hurd & Kuznetsov [23] to more general square-root diffusion processes. Note – in the setting of Lemma 2.5 – that if  $\mu(x) = \frac{\alpha}{x}$  for all  $x \in (0, \infty)$  and some  $\alpha \in (\frac{1}{2}, \infty)$ , then the argument of the exponential function in (28) is positive if  $p \in (0, 2\alpha - 1)$ .

**Lemma 2.5.** Assume the setting in Section 2.1 and assume  $X_t(\omega) \in (0, \infty)$  for all  $(t, \omega) \in [0, T] \times \Omega$ . Then it holds for all  $t \in [0, T]$ ,  $p \in \mathbb{R}$  that

$$\mathbb{E} \left[ (X_t)^{-p} \exp \left( p \int_0^t \frac{\mu(X_s)}{X_s} - \frac{p+1}{2(X_s)^2} ds \right) \right] \leq \mathbb{E}[(X_0)^{-p}]. \quad (28)$$

*Proof of Lemma 2.5.* First of all, we define a function  $\sigma: [0, \infty) \rightarrow \mathbb{R}$  by  $\sigma(x) = 1$  for all  $x \in [0, \infty)$  and we fix a real number  $p \in \mathbb{R}$ . Then we define a function  $U: (0, \infty) \rightarrow \mathbb{R}$  by  $U(x) := -p \log(x)$  for all  $x \in (0, \infty)$  and we note that  $U \in C^2((0, \infty), \mathbb{R})$ . Corollary 2.5 in Cox et. al [8] can thus be applied to obtain that for all  $t \in [0, T]$  it holds that

$$\mathbb{E} \left[ (X_t)^{-p} \exp \left( \int_0^t -(\mathcal{G}_{\mu, \sigma} U)(X_s) - \frac{1}{2} |(\nabla U)(X_s)|^2 ds \right) \right] \leq \mathbb{E}[(X_0)^{-p}]. \quad (29)$$

This together with the observation that for all  $x \in (0, \infty)$  it holds that  $\nabla U(x) = -\frac{p}{x}$  and  $(\mathcal{G}_{\mu, \sigma} U)(x) = -p \frac{\mu(x)}{x} + \frac{p}{2x^2}$  finishes the proof of Lemma 2.5.  $\square$

## 2.3 Temporal Hölder regularity based on Bessel-type processes

In the next two lemmas we present two elementary and essentially well-known estimates for SDEs with a globally one-sided Lipschitz continuous drift coefficient and an at most linearly growing diffusion coefficient.

**Lemma 2.6.** Assume the setting in Section 2.1, let  $c \in [0, \infty)$  and assume  $x\mu(x) \leq c(1 + x^2)$  for all  $x \in (0, \infty)$ . Then it holds for all  $q \in (0, \infty)$ ,  $t \in [0, T]$  that

$$\mathbb{E}[(X_t)^q] \leq \mathbb{E}[(1 + (X_t)^2)^{\frac{q}{2}}] \leq \exp \left( t [1 + (q-2)^+ + 2c] \right) \mathbb{E}[(1 + (X_0)^2)^{\frac{q}{2}}]. \quad (30)$$

*Proof of Lemma 2.6.* Observe that the estimate  $x\mu(x) \leq c(1 + x^2)$  for all  $x \in [0, \infty)$  implies that for all  $x \in [0, \infty)$ ,  $q \in (0, \infty)$  it holds that

$$2x\mu(x) + 1 + \frac{2 \left( \frac{q}{2} - 1 \right) x^2}{1 + x^2} \leq 1 + (q-2)^+ + 2c(1 + x^2) \leq (1 + (q-2)^+ + 2c) (1 + x^2). \quad (31)$$

This and Corollary 2.6 in Cox et. al [8] prove that for all  $q \in (0, \infty)$ ,  $t \in [0, T]$  it holds that

$$\mathbb{E}[(X_t)^q] \leq \mathbb{E}[(1 + (X_t)^2)^{\frac{q}{2}}] \leq \exp \left( t [1 + (q-2)^+ + 2c] \right) \mathbb{E}[(1 + (X_0)^2)^{\frac{q}{2}}]. \quad (32)$$

This finishes the proof of Lemma 2.6.  $\square$

**Lemma 2.7.** Assume the setting in Section 2.1, let  $c \in [0, \infty)$ ,  $q \in [2, \infty)$  and assume  $x\mu(x) \leq c(1 + x^2)$  for all  $x \in (0, \infty)$ . Then

$$\left\| \sup_{t \in [0, T]} (X_t)^2 \right\|_{L^q(\Omega; \mathbb{R})} \leq \left\| (X_0)^2 \right\|_{L^q(\Omega; \mathbb{R})} + 2cT \sup_{s \in [0, T]} \|X_s\|_{L^{2q}(\Omega; \mathbb{R})}^2 + \sqrt{\frac{2Tq^3}{q-1}} \sup_{s \in [0, T]} \|X_s\|_{L^q(\Omega; \mathbb{R})} + T(2c+1). \quad (33)$$

*Proof of Lemma 2.7.* First of all, observe that Itô's lemma and the assumption that for all  $x \in [0, \infty)$  it holds that  $x\mu(x) \leq c(1+x^2)$  implies that for all  $t \in [0, T]$  it holds that

$$\begin{aligned} (X_t)^2 &= (X_0)^2 + 2 \int_0^t X_s \mu(X_s) ds + t + 2 \int_0^t X_s dW_s \\ &\leq (X_0)^2 + 2c \int_0^t 1 + (X_s)^2 ds + t + 2 \int_0^t X_s dW_s. \end{aligned} \quad (34)$$

Furthermore, note that the Burkholder-Davis-Gundy-type inequalities in Lemma 7.2 and Lemma 7.7 in Da Prato & Zabczyk [9] show that

$$\left\| \sup_{t \in [0, T]} \int_0^t X_s dW_s \right\|_{L^q(\Omega; \mathbb{R})} \leq \sqrt{\frac{q^3}{2(q-1)} \int_0^T \|X_s\|_{L^q(\Omega; \mathbb{R})}^2 ds} \leq \sqrt{\frac{Tq^3}{2(q-1)}} \sup_{s \in [0, T]} \|X_s\|_{L^q(\Omega; \mathbb{R})}. \quad (35)$$

Combining (34) and (35) proves that

$$\begin{aligned} &\left\| \sup_{t \in [0, T]} (X_t)^2 \right\|_{L^q(\Omega; \mathbb{R})} \\ &\leq \|(X_0)^2\|_{L^q(\Omega; \mathbb{R})} + 2c \left\| \int_0^T (X_s)^2 ds \right\|_{L^q(\Omega; \mathbb{R})} + 2 \left\| \sup_{t \in [0, T]} \int_0^t X_s dW_s \right\|_{L^q(\Omega; \mathbb{R})} + T(2c+1) \\ &\leq \|(X_0)^2\|_{L^q(\Omega; \mathbb{R})} + 2cT \sup_{s \in [0, T]} \|(X_s)^2\|_{L^q(\Omega; \mathbb{R})} + \sqrt{\frac{2Tq^3}{q-1}} \sup_{s \in [0, T]} \|X_s\|_{L^q(\Omega; \mathbb{R})} + T(2c+1). \end{aligned} \quad (36)$$

This finishes the proof of Lemma 2.7.  $\square$

**Lemma 2.8.** Let  $T \in [0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $Z \in \mathcal{L}^0([0, T] \times \Omega; \mathbb{R})$  satisfy  $\int_0^T |Z_s| ds < \infty$   $\mathbb{P}$ -a.s. and let  $Y: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a stochastic process with continuous sample paths satisfying  $Y_t = \int_0^t Z_r dr$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . Then for all  $q \in (0, \infty)$ ,  $p \in [1, \infty)$  it holds that

$$\|Y\|_{C^{(1-1/p)}([0, T], \mathbb{R})} \Big\|_{L^q(\Omega; \mathbb{R})} \leq \|Z\|_{L^q(\Omega; L^p([0, T]; \mathbb{R}))} \quad (37)$$

and for all  $p, q \in (0, \infty)$ ,  $\delta \in [0, \infty]$ ,  $\theta \in [0, 1]$  satisfying  $q(p-\theta)(1+\delta) \geq p$  it holds that

$$\|Y\|_{C^{(1-\theta/p)}([0, T], L^q(\Omega; \mathbb{R}))} \leq \|Z\|_{L^{q\theta(1+1/\delta)}(\Omega; L^p([0, T]; \mathbb{R}))} \|Z\|_{L^\infty([0, T]; L^{q(1-\theta)(1+\delta)}(\Omega; \mathbb{R}))}^{(1-\theta)}. \quad (38)$$

*Proof of Lemma 2.8.* Hölder's inequality shows that for all  $q \in (0, \infty)$ ,  $p \in [1, \infty)$  it holds that

$$\begin{aligned} &\left\| \sup_{s, t \in [0, T]} \left[ |t-s|^{-[1-\frac{1}{p}]} \left| \int_s^t Z_r dr \right| \right] \right\|_{L^q(\Omega; \mathbb{R})} \\ &\leq \left\| \sup_{s, t \in [0, T]} \left[ |t-s|^{-[1-\frac{1}{p}]} \left( \int_s^t |Z_r|^p dr \right)^{\frac{1}{p}} \left( \int_s^t 1 dr \right)^{[1-\frac{1}{p}]} \right] \right\|_{L^q(\Omega; \mathbb{R})} = \left\| \int_0^T |Z_r|^p dr \right\|_{L^{q/p}(\Omega; \mathbb{R})}^{\frac{1}{p}}. \end{aligned} \quad (39)$$

This proves (37). Again Hölder's inequality shows that for all  $q \in (0, \infty)$ ,  $\delta \in [0, \infty]$ ,  $\theta \in (0, 1]$ ,  $p \in (\theta, \infty)$  satisfying  $q(p-\theta)(1+\delta) \geq p$  it holds that

$$\begin{aligned} &\sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \left\| \int_s^t Z_r dr \right\|_{L^q(\Omega; \mathbb{R})} \right] = \sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \left\| \int_s^t |Z_r|^\theta |Z_r|^{(1-\theta)} dr \right\|_{L^q(\Omega; \mathbb{R})} \right] \\ &\leq \sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \left\| \left( \int_s^t |Z_r|^p dr \right)^{\frac{\theta}{p}} \left( \int_s^t |Z_r|^{\frac{p(1-\theta)}{p-\theta}} dr \right)^{1-\frac{\theta}{p}} \right\|_{L^q(\Omega; \mathbb{R})} \right] \\ &\leq \sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \left\| \left( \int_s^t |Z_r|^p dr \right)^{\frac{\theta}{p}} \right\|_{L^{q(1+1/\delta)}(\Omega; \mathbb{R})} \left\| \left( \int_s^t |Z_r|^{\frac{p(1-\theta)}{p-\theta}} dr \right)^{1-\frac{\theta}{p}} \right\|_{L^{q(1+\delta)}(\Omega; \mathbb{R})} \right] \\ &= \sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \|Z\|_{L^{q\theta(1+1/\delta)}(\Omega; L^p([s, t]; \mathbb{R}))}^\theta \left\| \int_s^t |Z_r|^{\frac{p(1-\theta)}{p-\theta}} dr \right\|_{L^{\frac{q(p-\theta)(1+\delta)}{p}}(\Omega; \mathbb{R})}^{1-\frac{\theta}{p}} \right] \\ &\leq \sup_{\substack{s, t \in [0, T], \\ s \neq t}} \left[ |t-s|^{-[1-\frac{\theta}{p}]} \|Z\|_{L^{q\theta(1+1/\delta)}(\Omega; L^p([s, t]; \mathbb{R}))}^\theta \left( \int_s^t \|Z_r\|_{L^{q(1-\theta)(1+\delta)}(\Omega; \mathbb{R})}^{\frac{p(1-\theta)}{p-\theta}} dr \right)^{1-\frac{\theta}{p}} \right] \\ &\leq \|Z\|_{L^{q\theta(1+1/\delta)}(\Omega; L^p([0, T]; \mathbb{R}))}^\theta \|Z\|_{L^\infty([0, T]; L^{q(1-\theta)(1+\delta)}(\Omega; \mathbb{R}))}^{(1-\theta)}. \end{aligned} \quad (40)$$

This finishes the proof of Lemma 2.8.  $\square$

**Proposition 2.9** (Temporal Hölder regularity for Bessel-type processes). *Assume the setting in Section 2.1, let  $\alpha, q \in (0, \infty)$ ,  $\varepsilon \in (\frac{[\alpha-1/2]^+}{1+2\alpha}, \frac{2\alpha}{1+2\alpha})$ ,  $p = \frac{1+2\alpha}{\varepsilon(1+2\alpha)+1}$ ,  $c \in [p, \infty)$ ,  $\tilde{c} \in [0, \infty)$  and assume that for all  $x \in (0, \infty)$  it holds that  $\alpha - \tilde{c}x^c \leq x\mu(x) \leq \tilde{c}(1+x^2)$  and  $\mathbb{E}[(X_0)^{\max\{c-1, 1\} \max\{p, q, \frac{(q-p)(1+2\alpha+p)}{(1+2\alpha-p)}\}}] < \infty$ . Then*

$$\left\| \|X - W\|_{C^{(2\alpha/(1+2\alpha))-\varepsilon}([0, T], \mathbb{R})} \right\|_{L^q(\Omega; \mathbb{R})} \leq \|\mu(X)\|_{L^p([0, T]; \mathbb{R})} \| \cdot \|_{L^q(\Omega; \mathbb{R})} < \infty. \quad (41)$$

If, in addition to the above assumptions,  $\sup_{t \in [0, T]} \mathbb{E}[(X_t)^{-r}] < \infty$  for all  $r \in [0, (q/2p \wedge 1/2)(1+2\alpha+p)]$ , then

$$\begin{aligned} & \|X - W\|_{C^{((2\alpha(q+1)+1/q(1+2\alpha))-\varepsilon) \wedge 1}([0, T], L^q(\Omega; \mathbb{R}))} \\ & \leq \sup_{u \in [0, T]} \left\| \mu(X_u) \right\|_{L^{(q/2p \wedge 1/2)(1+2\alpha+p)}(\Omega; \mathbb{R})}^{p/q \wedge 1} \left\| \int_0^T |\mu(X_r)|^p dr \right\|_{L^{(q/p-1)((1+2\alpha+p)/(1+2\alpha-p))}(\Omega; \mathbb{R})}^{1/p-1/q} \in [0, \infty). \end{aligned} \quad (42)$$

*Proof of Proposition 2.9.* Lemma 2.6 and the assumption that

$$\mathbb{E} \left[ (X_0)^{\max\{c-1, 1\} \max\{p, q, \frac{(q-p)(1+2\alpha+p)}{(1+2\alpha-p)}\}} \right] < \infty \quad (43)$$

imply that

$$\sup_{t \in [0, T]} \mathbb{E} \left[ (X_t)^{\max\{c-1, 1\} \max\{p, q, \frac{(q-p)(1+2\alpha+p)}{(1+2\alpha-p)}\}} \right] < \infty. \quad (44)$$

Next we note that Jensen's inequality ensures that for all  $r \in (0, \infty)$  it holds that

$$\begin{aligned} & \mathbb{E} \left[ \left| \int_0^T (X_u)^{p \max\{c-1, 1\}} du \right|^r \right] \leq \mathbb{E} \left[ 1 + \left| \int_0^T (X_u)^{p((c-1) \vee 1)} du \right|^{r \vee 1} \right] \\ & = 1 + \mathbb{E} \left[ \left| \frac{1}{T} \int_0^T T (X_u)^{p((c-1) \vee 1)} du \right|^{r \vee 1} \right] \leq 1 + T^{[r-1]^+} \left[ \int_0^T \mathbb{E} \left[ (X_u)^{p((c-1) \vee 1)(r \vee 1)} \right] du \right]. \end{aligned} \quad (45)$$

In the next step we observe that the assumption that for all  $x \in (0, \infty)$  it holds that  $\alpha - \tilde{c}x^c \leq x\mu(x) \leq \tilde{c}(1+x^2)$  implies that for all  $x \in [0, \infty)$  it holds that  $|\mu(x)| \leq \frac{2(\alpha+\tilde{c})}{x}(1+x^{c \vee 2})$ . This, in turn, ensures that for all  $x \in [0, \infty)$  it holds that

$$|\mu(x)|^p \leq 4^p [\alpha + \tilde{c}]^p \left( \left[ \frac{1}{x} \right]^p + x^{p((c-1) \vee 1)} \right). \quad (46)$$

Lemma 2.3 together with the estimates (44), (45) and (46) implies that

$$\begin{aligned} & \left\| \int_0^T |\mu(X_r)|^p dr \right\|_{L^{\frac{q}{p} \vee \left[ \frac{(q-p)}{p} \frac{1+2\alpha+p}{1+2\alpha-p} \right]}(\Omega; \mathbb{R})} \leq 4^p [\alpha + \tilde{c}]^p \left\| \int_0^T \left[ \frac{1}{X_r} \right]^p dr \right\|_{L^{\frac{q}{p} \vee \left[ \frac{(q-p)}{p} \frac{1+2\alpha+p}{1+2\alpha-p} \right]}(\Omega; \mathbb{R})} \\ & + 4^p [\alpha + \tilde{c}]^p \left\| \int_0^T (X_r)^{p((c-1) \vee 1)} dr \right\|_{L^{\frac{q}{p} \vee \left[ \frac{(q-p)}{p} \frac{1+2\alpha+p}{1+2\alpha-p} \right]}(\Omega; \mathbb{R})} < \infty. \end{aligned} \quad (47)$$

Lemma 2.8 and (47) show that

$$\begin{aligned} & \left\| \sup_{s, t \in [0, T]} \left[ |t-s|^{-\frac{p-1}{p}} |X_t - X_s - W_t + W_s| \right] \right\|_{L^q(\Omega; \mathbb{R})} \\ & = \left\| \sup_{s, t \in [0, T]} \left[ |t-s|^{-\frac{p-1}{p}} \left| \int_s^t \mu(X_r) dr \right| \right] \right\|_{L^q(\Omega; \mathbb{R})} \leq \left\| \int_0^T |\mu(X_r)|^p dr \right\|_{L^{\frac{q}{p}}(\Omega; \mathbb{R})}^{\frac{1}{p}} < \infty. \end{aligned} \quad (48)$$

This together with the identity that  $\frac{p-1}{p} = \frac{2\alpha}{1+2\alpha} - \varepsilon \in (0, \frac{1}{2})$  proves (41). We define  $\theta := \left(1 - \frac{p}{q}\right) \vee 0 \in [0, 1]$  and  $\delta := \frac{1+2\alpha}{2p} - \frac{1}{2} \in (0, \infty)$ . Lemma 2.8 and  $q(p-\theta)(1+\delta) \geq p$  yield

$$\begin{aligned} & \sup_{s, t \in [0, T]} |t-s|^{-\frac{p-\theta}{p}} \left[ \|X_t - X_s - W_t + W_s\|_{L^q(\Omega; \mathbb{R})} \right] = \sup_{s, t \in [0, T]} \left[ |t-s|^{-\frac{p-\theta}{p}} \left\| \int_s^t \mu(X_r) dr \right\|_{L^q(\Omega; \mathbb{R})} \right] \\ & \leq \left\| \int_0^T |\mu(X_r)|^p dr \right\|_{L^{\frac{q\theta(1+\delta)}{p\delta}}(\Omega; \mathbb{R})}^{\frac{\theta}{p}} \sup_{u \in [0, T]} \|\mu(X_u)\|_{L^{q(1-\theta)(1+\delta)}(\Omega; \mathbb{R})}^{1-\theta}. \end{aligned} \quad (49)$$

Note that  $\frac{\theta}{p} = \left(\frac{1}{p} - \frac{1}{q}\right) \vee 0$ ,  $1-\theta = \frac{p}{q} \wedge 1$ ,  $1+\delta = \frac{1+2\alpha+p}{2p}$ ,  $q(1-\theta)(1+\delta) = \left(\frac{q}{p} \wedge 1\right) \frac{1+2\alpha+p}{2}$ . This together with (44) and  $\sup_{t \in [0, T]} \mathbb{E}[(X_t)^{-r}] < \infty$  for all  $r \in \left[0, \left(\frac{q}{p} \wedge 1\right) \frac{1+2\alpha+p}{2}\right]$  implies that

$$\sup_{u \in [0, T]} \|\mu(X_u)\|_{L^{q(1-\theta)(1+\delta)}(\Omega; \mathbb{R})}^{1-\theta} \leq 2(\alpha + \tilde{c}) \sup_{u \in [0, T]} \left\| \frac{1}{X_u} + X_u^{(c-1) \vee 1} \right\|_{L^{\left(\frac{q}{p} \wedge 1\right) \frac{1+2\alpha+p}{2}}(\Omega; \mathbb{R})}^{1-\theta} \in [0, \infty). \quad (50)$$



Furthermore, observe that  $\frac{1+\delta}{\delta} = 1 + \frac{2p}{1+2\alpha-p} = \frac{1+2\alpha+p}{1+2\alpha-p}$ ,  $\frac{q\theta(1+\delta)}{p\delta} = \left(\left(\frac{q}{p} - 1\right) \vee 0\right) \frac{1+2\alpha+p}{1+2\alpha-p}$  and that  $\frac{p-\theta}{p} = 1 - \left(\left(\frac{1}{p} - \frac{1}{q}\right) \vee 0\right) = \left(\frac{p-1}{p} + \frac{1}{q}\right) \wedge 1 = \left(\frac{2\alpha}{1+2\alpha} + \frac{1}{q} - \varepsilon\right) \wedge 1$ . This, (49), (47) and (50) imply that

$$\begin{aligned} & \|X - W\|_{C\left(\frac{2\alpha}{1+2\alpha} + \frac{1}{q} - \varepsilon\right) \wedge 1 ([0, T], L^q(\Omega; \mathbb{R}))} \\ & \leq \left\| \int_0^T |\mu(X_r)|^p dr \right\|_{L\left(\frac{q}{p} - 1\right) \frac{1+2\alpha+p}{1+2\alpha-p}(\Omega; \mathbb{R})}^{\frac{1}{p} - \frac{1}{q}} \sup_{u \in [0, T]} \left\| \mu(X_u) \right\|_{L\left(\frac{q}{p} \wedge 1\right) \frac{1+2\alpha+p}{2}(\Omega; \mathbb{R})}^{\frac{p}{q} \wedge 1} \in [0, \infty). \end{aligned} \quad (51)$$

This finishes the proof of Proposition 2.9.  $\square$

The next lemma presents an estimate that allows us to obtain Hölder-continuity of the mapping  $[0, T] \ni t \mapsto \frac{1}{X_t} \in L^1(\Omega; \mathbb{R})$  from Hölder-continuity of the mappings  $[0, T] \ni t \mapsto X_t \in L^q(\Omega; \mathbb{R})$ ,  $q \in (0, \infty)$ . In particular, in the setting of Section 2.1, this implies Hölder-continuity of the mapping  $[0, T] \ni t \mapsto \mu(X_t) \in L^1(\Omega; \mathbb{R})$  under certain assumptions. We also refer the reader to the literature on strong approximations of SDEs with discontinuous coefficients where different but related difficulties arise; see, e.g., [34], [14], [4] and [31].

**Lemma 2.10.** *Let  $T \in [0, \infty)$ ,  $c \in (0, \infty)$ ,  $f \in \mathcal{L}^0([0, \infty); \mathbb{R})$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X : [0, T] \times \Omega \rightarrow [0, \infty)$  be a stochastic process and assume  $\sup_{t \in [0, T]} \mathbb{P}[X_t = 0] = 0$  and  $|f(x) - f(y)| \leq c|x - y| \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} + x^c + y^c\right)$  for all  $x, y \in (0, \infty)$ . Then it holds for all  $\theta, \delta, q \in (0, \infty)$ ,  $\gamma \in (0, 1]$  that*

$$\begin{aligned} & \|f(X)\|_{C^{\gamma\theta}([0, T], L^q(\Omega; \mathbb{R}))} \leq c 7^{\max\{1, \frac{1}{q(1+\delta)}\}} \|X\|_{C^\theta([0, T], L^{q\gamma(1/\delta+1)}(\Omega; \mathbb{R}))}^\gamma \\ & \cdot \sup_{u \in [0, T]} \left[ 1 + \left\| \frac{1}{X_u} \right\|_{L^{q(1+\gamma)(1+\delta)}(\Omega; \mathbb{R})}^{1+\gamma} + \left\| X_u \right\|_{L^{q(1+\delta)(\max\{1/\gamma, c+1\}-\gamma)}(\Omega; \mathbb{R})}^{\max\{1/\gamma, c+1\}-\gamma} \right]. \end{aligned} \quad (52)$$

*Proof of Lemma 2.10.* First of all, observe that Young's inequality implies that for all  $x, y \in (0, \infty)$ ,  $\gamma \in (0, 1]$  it holds that

$$\begin{aligned} & (x+y)^{(1-\gamma)} \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \right) = \frac{x+y}{x(x+y)^\gamma} + \frac{x+y}{y(x+y)^\gamma} + \frac{x+y}{xy(x+y)^\gamma} = \frac{2+\frac{x}{y}+\frac{y}{x}}{(x+y)^\gamma} + \frac{\frac{1}{y}+\frac{1}{x}}{(x+y)^\gamma} \\ & \leq \frac{1}{x^\gamma} + \frac{1}{y^\gamma} + \frac{x^{1-\gamma}}{y} + \frac{y^{1-\gamma}}{x} + \frac{1}{y^{1+\gamma}} + \frac{1}{x^{1+\gamma}} \\ & \leq \frac{\gamma}{1+\gamma} \frac{1}{x^{1+\gamma}} + \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \frac{1}{y^{1+\gamma}} + \frac{1}{1+\gamma} + \frac{1}{1+\gamma} \frac{1}{y^{1+\gamma}} + \frac{\gamma}{1+\gamma} x^{\frac{1}{\gamma}-\gamma} + \frac{1}{1+\gamma} \frac{1}{x^{1+\gamma}} + \frac{\gamma}{1+\gamma} y^{\frac{1}{\gamma}-\gamma} + \frac{1}{1+\gamma} + \frac{1}{x^{1+\gamma}} \\ & = \frac{2}{x^{1+\gamma}} + \frac{2}{y^{1+\gamma}} + \frac{\gamma}{1+\gamma} x^{\frac{1}{\gamma}-\gamma} + \frac{\gamma}{1+\gamma} y^{\frac{1}{\gamma}-\gamma} + \frac{2}{1+\gamma}. \end{aligned} \quad (53)$$

Next note that the assumption that  $|f(x) - f(y)| \leq c|x - y| \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} + x^c + y^c\right)$  for all  $x, y \in (0, \infty)$ , the assumption  $\sup_{t \in [0, T]} \mathbb{P}[X_t = 0] = 0$ , Hölder's inequality and (53) show that for all  $t, s \in [0, T]$ ,  $\delta, q \in (0, \infty)$ ,  $\gamma \in (0, 1]$  it holds that

$$\begin{aligned} & \|f(X_t) - f(X_s)\|_{L^q(\Omega; \mathbb{R})} \\ & \leq c \left\| |X_t - X_s| \left( \frac{1}{X_t} + \frac{1}{X_s} + \frac{1}{X_t X_s} + (X_t)^c + (X_s)^c \right) \mathbb{1}_{\{X_t, X_s \in (0, \infty)\}} \right\|_{L^q(\Omega; \mathbb{R})} \\ & = c \left\| |X_t - X_s|^\gamma |X_t - X_s|^{(1-\gamma)} \left( \frac{1}{X_t} + \frac{1}{X_s} + \frac{1}{X_t X_s} + (X_t)^c + (X_s)^c \right) \mathbb{1}_{\{X_t, X_s \in (0, \infty)\}} \right\|_{L^q(\Omega; \mathbb{R})} \\ & \leq c \left\| |X_t - X_s|^\gamma \right\|_{L^{q(1+\frac{1}{\delta})}(\Omega; \mathbb{R})} \\ & \cdot \left\| |X_t - X_s|^{1-\gamma} \left( \frac{1}{X_t} + \frac{1}{X_s} + \frac{1}{X_t X_s} + (X_t)^c + (X_s)^c \right) \mathbb{1}_{\{X_t, X_s \in (0, \infty)\}} \right\|_{L^{q(1+\delta)}(\Omega; \mathbb{R})} \\ & \leq c \|X_t - X_s\|_{L^{q\gamma(1+\frac{1}{\delta})}(\Omega; \mathbb{R})}^\gamma \\ & \cdot \left\| \frac{2}{(X_t)^{1+\gamma}} + \frac{2}{(X_s)^{1+\gamma}} + \frac{\gamma}{1+\gamma} (X_t)^{\frac{1}{\gamma}-\gamma} + \frac{\gamma}{1+\gamma} (X_s)^{\frac{1}{\gamma}-\gamma} + \frac{2}{1+\gamma} + 2(X_t)^{c+1-\gamma} + 2(X_s)^{c+1-\gamma} \right\|_{L^{q(1+\delta)}(\Omega; \mathbb{R})}. \end{aligned} \quad (54)$$

Jensen's inequality hence shows that for all  $t, s \in [0, T]$ ,  $\delta, q \in (0, \infty)$ ,  $\gamma \in (0, 1]$  it holds that

$$\begin{aligned} & \|f(X_t) - f(X_s)\|_{L^q(\Omega; \mathbb{R})} \\ & \leq c 7^{\left[\frac{1}{q(1+\delta)} - 1\right]^+} \|X_t - X_s\|_{L^{q\gamma(1+\frac{1}{\delta})}(\Omega; \mathbb{R})}^\gamma \\ & \cdot \sup_{u \in [0, T]} \left[ \frac{2}{1+\gamma} + 4 \left\| \frac{1}{X_u} \right\|_{L^{q(1+\gamma)(1+\delta)}(\Omega; \mathbb{R})}^{1+\gamma} + \frac{2\gamma}{1+\gamma} \left\| (X_u)^{(1/\gamma-\gamma)} \right\|_{L^{q(1+\delta)}(\Omega; \mathbb{R})} + 4 \|X_u\|_{L^{q(1+\delta)(c+1-\gamma)}(\Omega; \mathbb{R})}^{c+1-\gamma} \right]. \end{aligned} \quad (55)$$

This implies that for all  $\theta, \delta, q \in (0, \infty)$ ,  $\gamma \in (0, 1]$  it holds that

$$\begin{aligned} & \sup_{s, t \in [0, T]} \left[ |t - s|^{-\gamma\theta} \|f(X_t) - f(X_s)\|_{L^q(\Omega; \mathbb{R})} \right] \\ & \leq c \cdot 7^{\left[\frac{1}{q(1+\delta)} - 1\right]^+} \cdot \sup_{s, t \in [0, T]} \left[ |t - s|^{-\theta} \|X_t - X_s\|_{L^{q\gamma(1+\frac{1}{\delta})}(\Omega; \mathbb{R})} \right]^\gamma \\ & \cdot \sup_{u \in [0, T]} \left[ \frac{2}{1+\gamma} + 4 \left\| \frac{1}{X_u} \right\|_{L^{q(1+\gamma)(1+\delta)}(\Omega; \mathbb{R})}^{1+\gamma} + \left[ \frac{2\gamma}{1+\gamma} + 4 \right] \max \left\{ 1, \left\| X_u \right\|_{L^{q(1+\delta)\left(\left(\frac{1}{\gamma} \vee (c+1)\right) - \gamma\right)}(\Omega; \mathbb{R})}^{\left(\frac{1}{\gamma} \vee (c+1)\right) - \gamma} \right\} \right]. \end{aligned} \quad (56)$$

This finishes the proof of Lemma 2.10.  $\square$

## 2.4 Temporal Hölder regularity based on Cox-Ingersoll-Ross type processes

**Lemma 2.11.** *Let  $T, x \in [0, \infty)$ ,  $(c_i)_{i \in \{1, 2, \dots, 7\}} \subseteq [0, \infty)$ ,  $\mu \in \mathcal{L}^0([0, \infty); \mathbb{R})$ ,  $\sigma \in \mathcal{L}^0([0, \infty); [0, \infty))$  satisfy  $\mu(z) \leq c_1 + c_2 z$  and  $|\sigma(z)|^2 \leq 2(c_3 z + c_4 z^2)$  for all  $z \in [0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying  $\int_0^t |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s. and*

$$X_t = x + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad (57)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . Then it holds for all  $t \in [0, T]$ ,  $p \in (0, \infty]$  that

$$\|X_t\|_{L^p(\Omega; \mathbb{R})} \leq e^{t(c_2 + c_4(p-1)^+)} \left[ x + t(c_1 + c_3(p-1)^+) \right]. \quad (58)$$

If, in addition to the above assumptions,  $|\mu(z)| \leq c_5 + c_6 z^{c_7}$  for all  $z \in [0, \infty)$ , then it holds for all  $t \in [0, T]$ ,  $p \in [1, \infty]$ ,  $q \in [p \vee 2, \infty]$  that

$$\begin{aligned} \|X_t - x\|_{L^p(\Omega; \mathbb{R})} &\leq c_5 t + e^{(c_7 \vee 1)t(c_2 + c_4(q(c_7 \vee 1) - 1))} \left( t c_6 \left[ x + t(c_1 + c_3(p c_7 - 1)^+) \right]^{c_7} \right. \\ &\quad \left. + \sqrt{q(q-1)} \sqrt{t c_3 x + \frac{1}{2} t^2 c_3 (c_1 + c_3(\frac{q}{2} - 1)) + t c_4 \left[ x + t(c_1 + c_3(q-1)) \right]^2} \right). \end{aligned} \quad (59)$$

*Proof of Lemma 2.11.* First of all, we show that for all  $\varepsilon \in (0, 1)$ ,  $k \in \mathbb{N}_0$ ,  $p \in (k, k+1]$ ,  $t \in [0, T]$  it holds that

$$\|X_t + \varepsilon\|_{L^p(\Omega; \mathbb{R})} \leq e^{t(c_2 + c_4(p-1)^+)} \left[ (x + \varepsilon) + t(c_1 + c_3(p-1)^+) \right] \quad (60)$$

by induction on  $k \in \mathbb{N}_0$ . For this observe that Itô's lemma implies that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} (X_t + \varepsilon)e^{-c_2 t} &= (x + \varepsilon) + \int_0^t (-c_2)(X_s + \varepsilon)e^{-c_2 s} + \mu(X_s)e^{-c_2 s} ds + \int_0^t \sigma(X_s)e^{-c_2 s} dW_s \\ &\leq (x + \varepsilon) + \int_0^t (-c_2)(X_s + \varepsilon)e^{-c_2 s} + (c_1 + c_2 X_s)e^{-c_2 s} ds + \int_0^t \sigma(X_s)e^{-c_2 s} dW_s \\ &\leq (x + \varepsilon) + \int_0^t c_1 e^{-c_2 s} ds + \int_0^t \sigma(X_s)e^{-c_2 s} dW_s \\ &\leq (x + \varepsilon) + c_1 t + \int_0^t \sigma(X_s)e^{-c_2 s} dW_s. \end{aligned} \quad (61)$$

This implies that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$  it holds that

$$\mathbb{E}[X_t + \varepsilon]e^{-c_2 t} \leq (x + \varepsilon) + c_1 t. \quad (62)$$

Note that inequality (60) in the case  $p \in (0, 1]$  follows immediately from (62). This proves the base case  $k = 0$  of (60). For the induction step we assume that (60) holds for some  $k \in \mathbb{N}_0$ . Then note with Itô's lemma that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $p \in (k+1, k+2]$  it holds that

$$\begin{aligned} (X_t + \varepsilon)^p e^{-pt(c_2 + c_4(p-1))} &= (x + \varepsilon)^p + \int_0^t -p(c_2 + c_4(p-1))(X_s + \varepsilon)^p e^{-ps(c_2 + c_4(p-1))} + p\mu(X_s)(X_s + \varepsilon)^{p-1} e^{-ps(c_2 + c_4(p-1))} \\ &\quad + \frac{p(p-1)}{2} |\sigma(X_s)|^2 (X_s + \varepsilon)^{p-2} e^{-ps(c_2 + c_4(p-1))} ds + p \int_0^t \sigma(X_s)(X_s + \varepsilon)^{p-1} e^{-ps(c_2 + c_4(p-1))} dW_s \\ &\leq (x + \varepsilon)^p + \int_0^t e^{-ps(c_2 + c_4(p-1))} \left[ -p(c_2 + c_4(p-1))(X_s + \varepsilon)^p + p(X_s + \varepsilon)^{p-1} (c_1 + c_2 X_s) \right. \\ &\quad \left. + p(p-1)(c_3 X_s + c_4(X_s)^2)(X_s + \varepsilon)^{p-2} \right] ds + p \int_0^t \sigma(X_s)(X_s + \varepsilon)^{p-1} e^{-ps(c_2 + c_4(p-1))} dW_s \\ &\leq (x + \varepsilon)^p + \int_0^t e^{-ps(c_2 + c_4(p-1))} p[c_1 + c_3(p-1)](X_s + \varepsilon)^{p-1} ds \\ &\quad + p \int_0^t \sigma(X_s)(X_s + \varepsilon)^{p-1} e^{-ps(c_2 + c_4(p-1))} dW_s. \end{aligned} \quad (63)$$

This implies that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $p \in (k+1, k+2]$  it holds that

$$\begin{aligned} \mathbb{E}[(X_t + \varepsilon)^p] e^{-pt(c_2 + c_4(p-1))} &\leq (x + \varepsilon)^p + \int_0^t e^{-ps(c_2 + c_4(p-1))} p[c_1 + c_3(p-1)] \mathbb{E}[(X_s + \varepsilon)^{p-1}] ds \\ &\leq (x + \varepsilon)^p + \int_0^t e^{-ps(c_2 + c_4(p-1))} p[c_1 + c_3(p-1)] e^{(p-1)s(c_2 + c_4(p-2)^+)} \\ &\quad \cdot \left[ (x + \varepsilon) + s(c_1 + c_3(p-2)^+) \right]^{p-1} ds \\ &\leq (x + \varepsilon)^p + p[c_1 + c_3(p-1)] \int_0^t \left[ (x + \varepsilon) + s(c_1 + c_3(p-1)) \right]^{p-1} ds \\ &= (x + \varepsilon)^p + \left[ \left( (x + \varepsilon) + t(c_1 + c_3(p-1)) \right)^p - (x + \varepsilon)^p \right] = \left( (x + \varepsilon) + t(c_1 + c_3(p-1)) \right)^p. \end{aligned} \quad (64)$$

Induction hence shows that inequality (60) holds for all  $k \in \mathbb{N}_0$ . Taking the infimum over all  $\varepsilon \in (0, 1)$  on the right-hand side of inequality (60) shows (58) in the case  $p \in (0, \infty)$ . The case  $p = \infty$  follows from letting  $(0, \infty) \ni p \rightarrow \infty$ . Now we assume in addition that for all  $z \in [0, \infty)$  it holds that  $|\mu(z)| \leq c_5 + c_6 z^{c_7}$ . Note that the Burkholder-Davis-Gundy-type inequality in Lemma 7.2 in Da Prato & Zabczyk [9], Jensen's inequality and (58) ensure that for all  $t \in [0, T]$ ,  $p \in [1, \infty)$ ,  $q \in [p \vee 2, \infty)$  it holds that

$$\begin{aligned}
\|X_t - x\|_{L^p(\Omega; \mathbb{R})} &\leq \left\| \int_0^t |\mu(X_s)| ds \right\|_{L^p(\Omega; \mathbb{R})} + \left\| \int_0^t \sigma(X_s) dW_s \right\|_{L^p(\Omega; \mathbb{R})} \\
&\leq \left\| \int_0^t |\mu(X_s)| ds \right\|_{L^p(\Omega; \mathbb{R})} + \left\| \int_0^t \sigma(X_s) dW_s \right\|_{L^q(\Omega; \mathbb{R})} \\
&\leq \int_0^t c_5 + c_6 \|(X_s)^{c_7}\|_{L^p(\Omega; \mathbb{R})} ds + \sqrt{\frac{q(q-1)}{2}} \sqrt{\int_0^t \|\sigma(X_s)\|^2\|_{L^{\frac{q}{2}}(\Omega; \mathbb{R})} ds} \\
&\leq \int_0^t c_5 + c_6 \|(X_s)^{c_7}\|_{L^p(\Omega; \mathbb{R})} ds + \sqrt{\frac{q(q-1)}{2}} \sqrt{2 \int_0^t c_3 \|X_s\|_{L^{\frac{q}{2}}(\Omega; \mathbb{R})} + c_4 \|X_s\|_{L^q(\Omega; \mathbb{R})}^2 ds} \\
&\leq c_5 t + c_6 \int_0^t e^{c_7 s(c_2 + c_4(p c_7 - 1)^+)} \left[ x + s(c_1 + c_3(p c_7 - 1)^+) \right]^{c_7} ds + \sqrt{q(q-1)} \\
&\quad \cdot \sqrt{\int_0^t c_3 e^{s(c_2 + c_4(q-1))} \left[ x + s(c_1 + c_3(\frac{q}{2} - 1)) \right] + c_4 e^{2s(c_2 + c_4(q-1))} \left[ x + s(c_1 + c_3(q-1)) \right]^2 ds}.
\end{aligned} \tag{65}$$

This shows that for all  $t \in [0, T]$ ,  $p \in [1, \infty]$ ,  $q \in [p \vee 2, \infty)$  it holds that

$$\begin{aligned}
\|X_t - x\|_{L^p(\Omega; \mathbb{R})} &\leq c_5 t + c_6 \int_0^t e^{c_7 s(c_2 + c_4(p c_7 - 1)^+)} \left[ x + s(c_1 + c_3(p c_7 - 1)^+) \right]^{c_7} ds + \sqrt{q(q-1)} \\
&\quad \cdot \sqrt{\int_0^t c_3 e^{s(c_2 + c_4(q-1))} \left[ x + s(c_1 + c_3(\frac{q}{2} - 1)) \right] + c_4 e^{2s(c_2 + c_4(q-1))} \left[ x + s(c_1 + c_3(q-1)) \right]^2 ds}.
\end{aligned} \tag{66}$$

Finally, (66) implies that for all  $t \in [0, T]$ ,  $p \in [1, \infty]$ ,  $q \in [p \vee 2, \infty]$  it holds that

$$\begin{aligned}
\|X_t - x\|_{L^p(\Omega; \mathbb{R})} &\leq c_5 t + e^{(c_7 \vee 1)t(c_2 + c_4(q(c_7 \vee 1) - 1))} \left( c_6 \int_0^t \left[ x + s(c_1 + c_3(p c_7 - 1)^+) \right]^{c_7} ds \right. \\
&\quad \left. + \sqrt{q(q-1)} \sqrt{\int_0^t c_3 \left[ x + s(c_1 + c_3(\frac{q}{2} - 1)) \right] + c_4 \left[ x + s(c_1 + c_3(q-1)) \right]^2 ds} \right) \\
&\leq c_5 t + e^{(c_7 \vee 1)t(c_2 + c_4(q(c_7 \vee 1) - 1))} \left( t c_6 \left[ x + t(c_1 + c_3(p c_7 - 1)^+) \right]^{c_7} \right. \\
&\quad \left. + \sqrt{q(q-1)} \sqrt{t c_3 x + \frac{1}{2} t^2 c_3 (c_1 + c_3(\frac{q}{2} - 1)) + t c_4 \left[ x + t(c_1 + c_3(q-1)) \right]^2} \right).
\end{aligned} \tag{67}$$

This finishes the proof of Lemma 2.11.  $\square$

**Lemma 2.12.** *Let  $T \in [0, \infty)$ ,  $(c_i)_{i \in \{1, 2, 3\}} \subseteq [0, \infty)$ ,  $\sigma \in \mathcal{L}^0([0, \infty); [0, \infty))$  satisfy  $\sigma(z) \geq \frac{c_1 \sqrt{z}}{1 + c_3 z^{c_2}}$  for all  $z \in [0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $X, Y : \Omega \rightarrow [0, \infty)$  be random variables. Then it holds for all  $p \in (0, \infty]$ ,  $r \in [0, \infty]$  that*

$$\begin{aligned}
&\left\| \int_Y^X \frac{1}{\sigma(z)} dz \right\|_{L^p(\Omega; \mathbb{R})} \\
&\leq \frac{2}{c_1} \|1 + c_3 X^{c_2} + c_3 Y^{c_2}\|_{L^{p(1+1/r)}(\Omega; \mathbb{R})} \min \left\{ \| |Y - X|^{1/2} \|_{L^{p(1+r)}(\Omega; \mathbb{R})}, \left\| \frac{|Y - X|}{\sqrt{Y} + \sqrt{X}} \right\|_{L^{p(1+r)}(\Omega; \mathbb{R})} \right\}.
\end{aligned} \tag{68}$$

*Proof of Lemma 2.12.* First of all, observe that for all  $v, w \in [0, \infty)$  it holds that

$$\begin{aligned}
\left| \int_w^v \frac{1}{\sigma(z)} dz \right| &\leq \frac{1}{c_1} \left| \int_w^v \frac{1}{\sqrt{z}} |1 + c_3 z^{c_2}| dz \right| \leq \frac{1}{c_1} \sup_{u \in [v, w] \cup [w, v]} |1 + c_3 u^{c_2}| \left| \int_w^v \frac{1}{\sqrt{z}} dz \right| \\
&\leq \frac{2}{c_1} |1 + c_3 v^{c_2} + c_3 w^{c_2}| |\sqrt{v} - \sqrt{w}| = \frac{2}{c_1} \frac{|v - w|}{\sqrt{v} + \sqrt{w}} |1 + c_3 v^{c_2} + c_3 w^{c_2}|.
\end{aligned} \tag{69}$$

From (69) and from the estimate  $|v - w| \leq (\sqrt{v} + \sqrt{w})^2$  for all  $v, w \in [0, \infty)$  we get that for all  $v, w \in [0, \infty)$  it holds that

$$\left| \int_w^v \frac{1}{\sigma(z)} dz \right|^2 \leq \frac{4}{(c_1)^2} |v - w| |1 + c_3 v^{c_2} + c_3 w^{c_2}|^2. \tag{70}$$

Inequality (69) and Hölder's inequality ensure that for all  $p \in (0, \infty]$ ,  $q \in [1, \infty]$  it holds that

$$\begin{aligned}
\left\| \int_X^Y \frac{1}{\sigma(z)} dz \right\|_{L^p(\Omega; \mathbb{R})} &\leq \frac{2}{c_1} \left\| \frac{Y - X}{\sqrt{Y} + \sqrt{X}} |1 + c_3 X^{c_2} + c_3 Y^{c_2}| \right\|_{L^p(\Omega; \mathbb{R})} \\
&\leq \frac{2}{c_1} \left\| \frac{Y - X}{\sqrt{Y} + \sqrt{X}} \right\|_{L^{pq}(\Omega; \mathbb{R})} \|1 + c_3 X^{c_2} + c_3 Y^{c_2}\|_{L^{p/(1-1/q)}(\Omega; \mathbb{R})}.
\end{aligned} \tag{71}$$

Similarly, inequality (70) and Hölder's inequality imply that for all  $p \in (0, \infty]$ ,  $q \in [1, \infty]$  it holds that

$$\begin{aligned} \left\| \int_Y^X \frac{1}{\sigma(z)} dz \right\|_{L^p(\Omega; \mathbb{R})} &= \left\| \left\| \int_Y^X \frac{1}{\sigma(z)} dz \right\|_{L^{p/2}(\Omega; \mathbb{R})}^2 \right\|_{L^{p/2}(\Omega; \mathbb{R})}^{1/2} \leq \left\| \frac{4}{(c_1)^2} |Y - X| |1 + c_3 X^{c_2} + c_3 Y^{c_2}|^2 \right\|_{L^{p/2}(\Omega; \mathbb{R})}^{1/2} \\ &\leq \frac{2}{c_1} \|Y - X\|_{L^{pq/2}(\Omega; \mathbb{R})}^{1/2} \left\| |1 + c_3 X^{c_2} + c_3 Y^{c_2}|^2 \right\|_{L^{p/(2-2/q)}(\Omega; \mathbb{R})}^{1/2} \\ &= \frac{2}{c_1} \|Y - X\|_{L^{pq/2}(\Omega; \mathbb{R})}^{1/2} \|1 + c_3 X^{c_2} + c_3 Y^{c_2}\|_{L^{p/(1-1/q)}(\Omega; \mathbb{R})}. \end{aligned} \quad (72)$$

This and (71) show that for all  $p \in (0, \infty]$ ,  $q \in [1, \infty]$  it holds that

$$\begin{aligned} &\left\| \int_Y^X \frac{1}{\sigma(z)} dz \right\|_{L^p(\Omega; \mathbb{R})} \\ &\leq \frac{2}{c_1} \|1 + c_3 X^{c_2} + c_3 Y^{c_2}\|_{L^{p/(1-1/q)}(\Omega; \mathbb{R})} \min \left\{ \|Y - X\|_{L^{pq/2}(\Omega; \mathbb{R})}^{1/2}, \left\| \frac{Y - X}{\sqrt{Y + \sqrt{X}}} \right\|_{L^{pq}(\Omega; \mathbb{R})} \right\}. \end{aligned} \quad (73)$$

This finishes the proof of Lemma 2.12.  $\square$

**Theorem 2.13.** Let  $T \in (0, \infty)$ ,  $(c_i)_{i \in \{1, 2, \dots, 10\}} \subseteq [0, \infty)$ ,  $\mu \in \mathcal{L}^0([0, \infty); \mathbb{R})$ ,  $\sigma \in \mathcal{L}^0([0, \infty), [0, \infty))$  satisfy  $c_8 > 0$ ,  $\mu(z) \leq c_1 + c_2 z$ ,  $|\sigma(z)|^2 \leq 2(c_3 z + c_4 z^2)$ ,  $|\mu(z)| \leq c_5 + c_6 z^{c_7}$ ,  $\sigma(z) \geq \frac{c_8 \sqrt{z}}{1 + c_{10} z^{c_9}}$  for all  $z \in [0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X^x: [0, T] \times \Omega \rightarrow [0, \infty)$ ,  $x \in [0, \infty)$ , be a family of adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \int_0^t \mu(X_s^x) ds + \int_0^t \sigma(X_s^x) dW_s \quad (74)$$

$\mathbb{P}$ -a.s. for all  $(t, x) \in [0, T] \times [0, \infty)$  and assume that for every compact set  $K \subseteq [0, \infty)$  it holds that

$$\sup_{\substack{x, y \in K, \\ x \neq y}} \frac{(x-y)(\mu(x)-\mu(y))}{|x-y|^2} + \sup_{\substack{x, y \in K, \\ x \neq y}} \frac{(\sigma(x)-\sigma(y))^2}{|x-y|} < \infty. \quad (75)$$

Then it holds for all  $s, t \in [0, T]$ ,  $x \in [0, \infty)$ ,  $p \in (0, \infty)$ ,  $q \in [1, \infty)$  with  $pq \geq 2$  that

$$\begin{aligned} &\left\| \int_{X_s^x}^{X_t^x} \frac{1}{\sigma(z)} dz \right\|_{L^p(\Omega; \mathbb{R})} \leq \sqrt{|t-s|} \\ &\cdot \frac{2}{c_8} \left[ 1 + e^{(t \wedge s)(c_2 + c_4[p(c_9 + \max\{c_7, 1/2\}) - 1]^+)} \left[ x + (t \wedge s)(c_1 + c_3(p(c_9 + (c_7 \vee \frac{1}{2}))) - 1)^+ \right] \right]^{c_9 + (c_7 \vee 1/2)} \\ &\cdot \left[ 1 + c_{10} + c_{10} e^{|t-s| c_9 (c_2 + c_4 [\frac{c_9 pq}{q-1} - 1]^+)} \left[ 1 + |t-s| \left( c_1 + c_3 \left[ \frac{c_9 pq}{q-1} - 1 \right]^+ \right) \right]^{c_9} \right] \\ &\cdot \max \left\{ 1, c_5 + e^{(c_7 \vee 1)|t-s|(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ 1 + |t-s|(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\ &\quad \left. \left. + \sqrt{pq(pq-1)} \sqrt{c_3 + \frac{c_1 c_3}{2} + \frac{c_2^2}{2} \left( \frac{pq}{2} - 1 \right) + c_4 \left[ 1 + \sqrt{|t-s|(c_1 + c_3(pq-1))} \right]^2} \right) \right\} \end{aligned} \quad (76)$$

and it holds for all  $p, \varepsilon \in (0, \infty)$  that

$$\sup_{x \in [0, \infty)} \left[ \frac{1}{1 + x^{c_9 + \max\{c_7, 1/2\}}} \left\| \sup_{s, t \in [0, T], s \neq t} \left[ |t-s|^{(\varepsilon-1/2)} \left| \int_{X_s^x}^{X_t^x} \frac{1}{\sigma(z)} dz \right| \right] \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \quad (77)$$

*Proof of Theorem 2.13.* From Theorem V.40.1 in [33] together with (75) (the proof of Theorem V.40.1 in [33] only uses global one-sided Lipschitz continuity of  $\mu$ ) and from a stopping argument, we obtain that the SDE with coefficients  $(\mu, \sigma)$  is pathwise unique. Then we fix  $t_1 \in [0, T)$  and define  $W^{t_1}: [0, T - t_1] \times \Omega \rightarrow \mathbb{R}$  by  $W_t^{t_1} := W_{t_1+t} - W_{t_1}$  for all  $t \in [0, T - t_1]$ . Theorem 3.4 in Cox et. al [8] shows that there exists a measurable mapping  $Y: [0, T - t_1] \times [0, \infty) \times \Omega \rightarrow [0, \infty)$  such that for all  $x \in [0, \infty)$  it holds that  $Y^x$  is an adapted stochastic process with continuous sample paths satisfying

$$Y_t^x = x + \int_0^t \mu(Y_s^x) ds + \int_0^t \sigma(Y_s^x) dW_s^{t_1} \quad (78)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T - t_1]$ . Pathwise uniqueness ensures that for all  $x \in [0, \infty)$  it holds that

$$\mathbb{P}[(X_{t_1+t}^x)_{t \in [0, T-t_1]} \in \cdot] = \mathbb{P}[(Y_t^{X_{t_1}^x})_{t \in [0, T-t_1]} \in \cdot]. \quad (79)$$

Lemma 2.11 shows that for all  $t \in [0, T - t_1]$ ,  $x \in [0, t]$ ,  $p, q \in (0, \infty)$  with  $pq \geq 2$  it holds that

$$\begin{aligned}
& \|Y_t^x - x\|_{L^{pq/2}(\Omega; \mathbb{R})} \leq \|Y_t^x - x\|_{L^{pq}(\Omega; \mathbb{R})} \\
& \leq c_5 t + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( tc_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \\
& \quad \left. + \sqrt{pq(pq - 1)} \sqrt{tc_3 x + \frac{1}{2}t^2 c_3 (c_1 + c_3(\frac{pq}{2} - 1)) + tc_4 \left[ x + t(c_1 + c_3(pq - 1)) \right]^2} \right) \\
& \leq c_5 t + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( tc_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \\
& \quad \left. + \sqrt{pq(pq - 1)} \sqrt{c_3 t^2 + \frac{1}{2}t^2 c_3 (c_1 + c_3(\frac{pq}{2} - 1)) + tc_4 \left[ \sqrt{t}\sqrt{x} + t(c_1 + c_3(pq - 1)) \right]^2} \right) \\
& = t \left[ c_5 + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\
& \quad \left. \left. + \sqrt{pq(pq - 1)} \sqrt{c_3 + \frac{c_3}{2} (c_1 + c_3(\frac{pq}{2} - 1)) + c_4 \left[ \sqrt{x} + \sqrt{t}(c_1 + c_3(pq - 1)) \right]^2} \right) \right].
\end{aligned} \tag{80}$$

Similarly, Lemma 2.11 shows that for all  $t \in [0, T - t_1]$ ,  $x \in [t, \infty)$ ,  $p, q \in (0, \infty)$  with  $pq \geq 2$  it holds that

$$\begin{aligned}
& \frac{1}{\sqrt{x}} \|Y_t^x - x\|_{L^{pq}(\Omega; \mathbb{R})} \\
& \leq c_5 \frac{t}{\sqrt{x}} + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( \frac{t}{\sqrt{x}} c_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \\
& \quad \left. + \sqrt{pq(pq - 1)} \sqrt{tc_3 + \frac{t^2 c_3}{2x} (c_1 + c_3(\frac{pq}{2} - 1)) + \frac{tc_4}{x} \left[ x + t(c_1 + c_3(pq - 1)) \right]^2} \right) \\
& \leq c_5 \sqrt{t} + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( \sqrt{t} c_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \\
& \quad \left. + \sqrt{pq(pq - 1)} \sqrt{tc_3 + \frac{tc_3}{2} (c_1 + c_3(\frac{pq}{2} - 1)) + tc_4 \left[ \sqrt{x} + \sqrt{t}(c_1 + c_3(pq - 1)) \right]^2} \right) \\
& = \sqrt{t} \left[ c_5 + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\
& \quad \left. \left. + \sqrt{pq(pq - 1)} \sqrt{c_3 + \frac{c_3}{2} (c_1 + c_3(\frac{pq}{2} - 1)) + c_4 \left[ \sqrt{x} + \sqrt{t}(c_1 + c_3(pq - 1)) \right]^2} \right) \right].
\end{aligned} \tag{81}$$

Next we define a function  $\phi: [0, \infty) \rightarrow [0, \infty)$  through  $\phi(y) := \int_0^y \frac{1}{\sigma(z)} dz$  for all  $y \in [0, \infty)$ . The function  $\phi$  is well defined due to the estimate that  $\sigma(z) \geq \frac{c_8 \sqrt{z}}{1 + c_{10} z^{c_9}}$  for all  $z \in [0, \infty)$ . In the next step we observe that Lemma 2.12, Lemma 2.11, (80) and (81) imply that for all  $t \in [0, T - t_1]$ ,  $x \in [0, \infty)$ ,  $p \in (0, \infty)$ ,  $q \in [1, \infty)$  with  $pq \geq 2$  it holds that

$$\begin{aligned}
& \|\phi(Y_t^x) - \phi(x)\|_{L^p(\Omega; \mathbb{R})} \\
& \leq \frac{2}{c_8} \left( 1 + c_{10} x^{c_9} + c_{10} \|Y_t^x\|_{L^{c_9 pq/q-1}(\Omega; \mathbb{R})}^{c_9} \right) \min \left\{ \|Y_t^x - x\|_{L^{pq/2}(\Omega; \mathbb{R})}^{1/2}, \frac{1}{\sqrt{x}} \|Y_t^x - x\|_{L^{pq}(\Omega; \mathbb{R})} \right\} \\
& \leq \frac{2}{c_8} \left( 1 + c_{10} x^{c_9} + c_{10} e^{tc_9(c_2 + c_4(\frac{c_9 pq}{q-1} - 1)^+)} \left[ x + t \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \\
& \quad \cdot \min \left\{ \|Y_t^x - x\|_{L^{pq/2}(\Omega; \mathbb{R})}^{1/2}, \frac{1}{\sqrt{x}} \|Y_t^x - x\|_{L^{pq}(\Omega; \mathbb{R})} \right\} \\
& \leq \frac{2}{c_8} \sqrt{t} \left( 1 + c_{10} x^{c_9} + c_{10} e^{tc_9(c_2 + c_4(\frac{c_9 pq}{q-1} - 1)^+)} \left[ x + t \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \\
& \quad \cdot \left\{ \left[ c_5 + e^{(c_7 \vee 1)t(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ x + t(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \right. \\
& \quad \left. \left. + \sqrt{pq(pq - 1)} \sqrt{c_3 + \frac{c_3}{2} (c_1 + c_3(\frac{pq}{2} - 1)) + c_4 \left[ \sqrt{x} + \sqrt{t}(c_1 + c_3(pq - 1)) \right]^2} \right) \right] \vee 1 \right\}.
\end{aligned} \tag{82}$$

Then (79) and (82) yield that for all  $h \in [0, T - t_1]$ ,  $x \in [0, \infty)$ ,  $p \in (0, \infty)$ ,  $q \in [1, \infty)$  with  $pq \geq 2$  it holds that

$$\begin{aligned} \|\phi(X_{t_1+h}^x) - \phi(X_{t_1}^x)\|_{L^p(\Omega; \mathbb{R})} &= \left( \mathbb{E} \left[ \mathbb{E} \left[ \left( \phi(Y_h^{X_{t_1}^x}) - \phi(X_{t_1}^x) \right)^p \mid X_{t_1}^x \right] \right] \right)^{1/p} \\ &\leq \left\| \frac{2}{c_8} \sqrt{h} \left( 1 + c_{10} (X_{t_1}^x)^{c_9} + c_{10} e^{hc_9 \left( c_2 + c_4 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right)} \left[ X_{t_1}^x + h \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \right. \\ &\quad \cdot \left[ c_5 + e^{(c_7 \vee 1)h(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ X_{t_1}^x + h(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\ &\quad \left. \left. + \sqrt{pq(pq-1)} \sqrt{c_3 + \frac{c_1 c_3}{2} + \frac{c_3^2}{2} \left( \frac{pq}{2} - 1 \right) + c_4 \left[ \sqrt{X_{t_1}^x} + \sqrt{h}(c_1 + c_3 pq - c_3) \right]^2} \right) \right] \vee 1 \right\|_{L^p(\Omega; \mathbb{R})} \end{aligned} \quad (83)$$

and

$$\begin{aligned} \|\phi(X_{t_1+h}^x) - \phi(X_{t_1}^x)\|_{L^p(\Omega; \mathbb{R})} &\leq \left\| \frac{2}{c_8} \sqrt{h} (1 + X_{t_1}^x)^{c_9 + (c_7 \vee \frac{1}{2})} \right. \\ &\quad \cdot \left( 1 + c_{10} + c_{10} e^{hc_9 \left( c_2 + c_4 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right)} \left[ 1 + h \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \\ &\quad \cdot \left[ c_5 + e^{(c_7 \vee 1)h(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ 1 + h(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\ &\quad \left. \left. + \sqrt{pq(pq-1)} \sqrt{c_3 + \frac{c_1 c_3}{2} + \frac{c_3^2}{2} \left( \frac{pq}{2} - 1 \right) + c_4 \left[ 1 + \sqrt{h}(c_1 + c_3 pq - c_3) \right]^2} \right) \right] \vee 1 \right\|_{L^p(\Omega; \mathbb{R})} \\ &= \frac{2}{c_8} \sqrt{h} \left\| 1 + X_{t_1}^x \right\|_{L^{p(c_9 + (c_7 \vee 1/2))}(\Omega; \mathbb{R})}^{c_9 + (c_7 \vee \frac{1}{2})} \\ &\quad \cdot \left( 1 + c_{10} + c_{10} e^{hc_9 \left( c_2 + c_4 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right)} \left[ 1 + h \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \\ &\quad \cdot \left[ c_5 + e^{(c_7 \vee 1)h(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ 1 + h(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\ &\quad \left. \left. + \sqrt{pq(pq-1)} \sqrt{c_3 + \frac{c_1 c_3}{2} + \frac{c_3^2}{2} \left( \frac{pq}{2} - 1 \right) + c_4 \left[ 1 + \sqrt{h}(c_1 + c_3 pq - c_3) \right]^2} \right) \right] \vee 1. \end{aligned} \quad (84)$$

This and Lemma 2.11 yield that for all  $h \in [0, T - t_1]$ ,  $x \in [0, \infty)$ ,  $p \in (0, \infty)$ ,  $q \in [1, \infty)$  with  $pq \geq 2$  it holds that

$$\begin{aligned} \|\phi(X_{t_1+h}^x) - \phi(X_{t_1}^x)\|_{L^p(\Omega; \mathbb{R})} &\leq \frac{2\sqrt{h}}{c_8} \left[ 1 + e^{t_1(c_2 + c_4(p(c_9 + (c_7 \vee 1/2)) - 1)^+)} \left[ x + t_1(c_1 + c_3(p(c_9 + (c_7 \vee 1/2)) - 1)^+) \right] \right]^{c_9 + (c_7 \vee \frac{1}{2})} \\ &\quad \cdot \left( 1 + c_{10} + c_{10} e^{hc_9 \left( c_2 + c_4 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right)} \left[ 1 + h \left( c_1 + c_3 \left( \frac{c_9 pq}{q-1} - 1 \right)^+ \right) \right]^{c_9} \right) \\ &\quad \cdot \left[ c_5 + e^{(c_7 \vee 1)h(c_2 + c_4(pq(c_7 \vee 1) - 1))} \left( c_6 \left[ 1 + h(c_1 + c_3(pqc_7 - 1)^+) \right]^{c_7} \right. \right. \\ &\quad \left. \left. + \sqrt{pq(pq-1)} \sqrt{c_3 + \frac{c_1 c_3}{2} + \frac{c_3^2}{2} \left( \frac{pq}{2} - 1 \right) + c_4 \left[ 1 + \sqrt{h}(c_1 + c_3 pq - c_3) \right]^2} \right) \right] \vee 1. \end{aligned} \quad (85)$$

This, in particular, proves that for all  $p \in (0, \infty)$ ,  $q \in [1, \infty)$  with  $pq \geq 2$  it holds that

$$\sup_{x \in [0, \infty)} \left[ \frac{1}{[1 + x^{c_9 + \max\{c_7, 1/2\}}]} \left[ \sup_{t_1, t_2 \in [0, T], t_1 \neq t_2} \frac{\|\phi(X_{t_2}^x) - \phi(X_{t_1}^x)\|_{L^p(\Omega; \mathbb{R})}}{|t_2 - t_1|^{1/2}} \right] \right] < \infty \quad (86)$$

Combining this with the Sobolev embedding that for all  $p \in (1, \infty)$ ,  $\alpha \in (1/p, 1]$ ,  $\varepsilon \in (0, \alpha - 1/p)$  it holds that  $C^\alpha([0, T], L^p(\Omega; \mathbb{R})) \subseteq L^p(\Omega; C^{\alpha-1/p-\varepsilon}([0, T], \mathbb{R}))$  continuously implies that for all  $p, \varepsilon \in (0, \infty)$  it holds that

$$\sup_{x \in [0, \infty)} \left[ \frac{1}{1 + x^{c_9 + \max\{c_7, 1/2\}}} \left\| \sup_{s, t \in [0, T], s \neq t} \frac{|\phi(X_t^x) - \phi(X_s^x)|}{|t - s|^{1/2-\varepsilon}} \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \quad (87)$$

This finishes the proof of Theorem 2.13.  $\square$

The following corollary, Corollary 2.14, specialises Theorem 2.13 (applied with  $c_2 = c_4 = c_9 = c_{10} = 0$ ,  $c_1 = \delta$ ,  $c_3 = \frac{\beta^2}{2}$ ,  $c_5 = \delta$ ,  $c_6 = \gamma^+$ ,  $c_7 = 1$ ,  $c_8 = \beta$ ,  $q = 1$ ) for Cox-Ingersoll-Ross processes and generalizes Lemma 3.2 in Dereich, Neuenkirch & Szpruch [11] which assumes  $\frac{2\delta}{\beta^2} \in (1, \infty)$ .

**Corollary 2.14.** *Let  $\delta \in [0, \infty)$ ,  $\gamma \in \mathbb{R}$ ,  $T, \beta \in (0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion and let  $X^x: [0, T] \times \Omega \rightarrow [0, \infty)$ ,  $x \in [0, \infty)$ , be a family of adapted stochastic processes with continuous sample paths satisfying*

$$X_t^x = x + \int_0^t \delta - \gamma X_s^x ds + \int_0^t \beta \sqrt{X_s^x} dW_s \quad (88)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ ,  $x \in [0, \infty)$ . Then it holds for all  $s, t \in [0, T]$ ,  $x \in [0, \infty)$ ,  $p \in (0, \infty)$  that

$$\begin{aligned} \left\| \sqrt{X_t^x} - \sqrt{X_s^x} \right\|_{L^p(\Omega; \mathbb{R})} &\leq \sqrt{|t-s|} \left[ 1 + x + \min\{s, t\} \left[ \delta + \frac{\beta^2}{2} (p-1)^+ \right] \right] \\ &\cdot \max \left\{ 1, \delta + \gamma^+ \left[ 1 + |t-s| \left[ \delta + \frac{\beta^2}{2} \left[ \frac{p}{2} - 1 \right]^+ \right] \right] + \beta \sqrt{\max(p(p-1), 2)} \sqrt{1 + \frac{1}{2} \left[ \delta + \frac{\beta^2}{2} \max\{p-1, 1\} \right]} \right\} \end{aligned} \quad (89)$$

and it holds for all  $p, \varepsilon \in (0, \infty)$  that

$$\sup_{x \in [0, \infty)} \left[ \frac{1}{1+x} \left\| \sup_{s, t \in [0, T], s \neq t} \frac{|\sqrt{X_t^x} - \sqrt{X_s^x}|}{|t-s|^{1/2-\varepsilon}} \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \quad (90)$$

### 3 Strong convergence rates for drift-implicit (square-root) Euler approximations of Bessel- and Cox-Ingersoll-Ross-type processes

Strong convergence rates are established in Corollary 3.9 and Theorem 3.13 below.

#### 3.1 On the relation between Bessel- and Cox-Ingersoll-Ross-type processes

In Lemma 3.2 below we establish, roughly speaking, that a transformation of a solution process of an SDE with non-additive noise is a solution process of a suitable SDE with additive noise (cf., e.g., Exercise XI.1.26 in Revuz & Yor [32] for the case of CIR processes). In the proof of Lemma 3.2 the following result, Lemma 3.1, is used.

**Lemma 3.1** (Time at non-trap boundaries has Lebesgue measure zero). *Let  $I \subseteq \mathbb{R}$  be an open interval, let  $T \in [0, \infty)$ ,  $J \subseteq \overline{I}$ ,  $\mu \in \mathcal{L}^0(J; \mathbb{R})$ ,  $\sigma \in \mathcal{L}^0(J, [0, \infty))$  satisfy  $I \subseteq J$  and  $\sup_{z \in K} |\sigma(z)| < \infty$  for all compact sets  $K \subseteq J$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow J$  be an adapted stochastic process with continuous sample paths satisfying  $\int_0^T |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s. and*

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad (91)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . Then it holds for all  $b \in \{z \in J: |\mu(z)| > 0, \limsup_{J \setminus \{z\} \ni x \rightarrow z} \frac{(\sigma \cdot \sigma)(x)}{|x-z|} < \infty\}$  that

$$\mathbb{P} \left[ \int_0^T \mathbb{1}_{\{X_s=b\}} ds = 0 \right] = 1. \quad (92)$$

*Proof of Lemma 3.1.* Define a set  $\Gamma := \{z \in J: |\mu(z)| > 0, \limsup_{J \setminus \{z\} \ni x \rightarrow z} \frac{(\sigma \cdot \sigma)(x)}{|x-z|} < \infty\}$  and define a family  $f_n: \mathbb{R} \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , of functions by

$$f_n(x) := \begin{cases} x \exp\left(1 - \frac{1}{1-n^2 x^2}\right) & : x \in (-\frac{1}{n}, \frac{1}{n}) \\ 0 & : x \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}) \end{cases} \quad (93)$$

for all  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . Observe that for all  $n \in \mathbb{N}$  it holds that  $f_n \in C^2(\mathbb{R}, \mathbb{R})$  and note that for all  $n \in \mathbb{N}$ ,  $x \in \mathbb{R}$  it holds that

$$f'_n(x) = \begin{cases} \exp\left(1 - \frac{1}{1-n^2 x^2}\right) - 2n^2 x^2 \frac{\exp\left(1 - \frac{1}{1-n^2 x^2}\right)}{(1-n^2 x^2)^2} & : x \in (-\frac{1}{n}, \frac{1}{n}) \\ 0 & : x \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}) \end{cases} \quad \text{and} \quad (94)$$

$$f''_n(x) = \begin{cases} -\frac{6n^2 x \exp\left(1 - \frac{1}{1-n^2 x^2}\right)}{(1-n^2 x^2)^2} - \frac{8n^4 x^3 \exp\left(1 - \frac{1}{1-n^2 x^2}\right)}{(1-n^2 x^2)^3} + \frac{4n^4 x^3 \exp\left(1 - \frac{1}{1-n^2 x^2}\right)}{(1-n^2 x^2)^4} & : x \in (-\frac{1}{n}, \frac{1}{n}) \\ 0 & : x \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}). \end{cases} \quad (95)$$

Next observe that (93) and (94) ensure that for all  $z \in \mathbb{R}$  it holds that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |f_n(x)| = 0, \quad \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |f'_n(x)| \leq 3 \exp(1) \quad \text{and} \quad \lim_{n \rightarrow \infty} f'_n(z) = \mathbb{1}_{\{0\}}(z). \quad (96)$$

Furthermore, note that (95) shows that

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |f_n''(x)x| \leq \sup_{x \in (-1,1)} \frac{18 \exp\left(\frac{1-\frac{1}{1-x^2}}{(1-x^2)^4}\right)}{(1-x^2)^4} < \infty. \quad (97)$$

Next define a family  $\phi_b: J \rightarrow [0, \infty)$ ,  $b \in \Gamma$ , of functions by

$$\phi_b(z) := \begin{cases} \frac{(\sigma \cdot \sigma)(z)}{|z-b|} & : z \in J \setminus \{b\} \\ \limsup_{J \setminus \{b\} \ni x \rightarrow b} \frac{(\sigma \cdot \sigma)(x)}{|x-b|} & : z \in \{b\} \end{cases} \quad (98)$$

for all  $z \in J$ ,  $b \in \Gamma$ . The boundedness of  $\sigma$  on compact subsets of  $J$  and the fact that for all  $b \in \Gamma$  it holds that  $\limsup_{J \setminus \{b\} \ni x \rightarrow b} \frac{(\sigma \cdot \sigma)(x)}{|x-b|} < \infty$  implies that for every  $b \in \Gamma$  and every compact set  $K \subseteq J$  it holds that  $\phi_b|_K$  is an upper semi-continuous function on the compact set  $K$  and, therefore, that  $\phi_b|_K$  is bounded from above. Furthermore, note that (97), (98) and the fact that for all  $n \in \mathbb{N}$  it holds that  $f_n''(0) = 0$  show that for all  $b \in \Gamma$  and all compact sets  $K \subseteq J$  it holds that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sup_{z \in K} |f_n''(z-b)(\sigma \cdot \sigma)(z)| &= \sup_{n \in \mathbb{N}} \sup_{z \in K \setminus \{b\}} \left| \frac{(z-b)f_n''(z-b)(\sigma \cdot \sigma)(z)}{z-b} \right| \\ &\leq \left[ \sup_{z \in K} \phi_b(z) \right] \left[ \sup_{z \in (-1,1)} \frac{18 \exp\left(\frac{1-\frac{1}{1-z^2}}{(1-z^2)^4}\right)}{(1-z^2)^4} \right] < \infty. \end{aligned} \quad (99)$$

In the next step we define stopping times  $\tau_k: \Omega \rightarrow [0, T]$ ,  $k \in \mathbb{N}$ , by

$$\tau_k := \inf \left( \{T\} \cup \left\{ t \in [0, T] : \int_0^t |\mu(X_s)| ds + |X_t| > k \right\} \cup \left\{ t \in [0, T] : (\exists b \in (\partial I) \setminus J : |X_t - b| < \frac{1}{k}) \right\} \right) \quad (100)$$

for all  $k \in \mathbb{N}$ . Itô's lemma implies that

$$\begin{aligned} f_n(X_{t \wedge \tau_k} - b) - f_n(X_0 - b) &= \int_0^{t \wedge \tau_k} f_n'(X_s - b) \mu(X_s) + \frac{1}{2} f_n''(X_s - b) (\sigma \cdot \sigma)(X_s) ds \\ &\quad + \int_0^{t \wedge \tau_k} f_n'(X_s - b) \sigma(X_s) dW_s \end{aligned} \quad (101)$$

$\mathbb{P}$ -a.s. for all  $b \in \Gamma$ ,  $t \in [0, T]$ ,  $n, k \in \mathbb{N}$ . Next note that the boundedness of  $\sigma$  on compact subsets of  $J$ , the compactness of the sets  $K_k := [-k, k] \cap \{z \in J : (\forall b \in (\partial I) \setminus J : |z - b| \geq \frac{1}{k})\} \subseteq J$ ,  $k \in \mathbb{N}$ , and (96) imply that for all  $b \in \Gamma$ ,  $k \in \mathbb{N}$  it holds that

$$\sup_{n \in \mathbb{N}} \sup_{x \in K_k} [f_n'(x-b)\sigma(x)]^2 \leq \left[ \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |f_n'(x)| \right]^2 \left[ \sup_{x \in K_k} |\sigma(x)| \right]^2 < \infty. \quad (102)$$

This implies that the expectation of the stochastic integral on the right-hand side of (101) vanishes. Hence, it holds for all  $b \in \Gamma$ ,  $t \in [0, T]$ ,  $n, k \in \mathbb{N}$  that

$$\mathbb{E}[f_n(X_{t \wedge \tau_k} - b) - f_n(X_0 - b)] = \mathbb{E} \left[ \int_0^{t \wedge \tau_k} f_n'(X_s - b) \mu(X_s) + \frac{1}{2} f_n''(X_s - b) (\sigma \cdot \sigma)(X_s) ds \right]. \quad (103)$$

The dominated convergence theorem together with (96), (99) and (95) shows that for all  $b \in \Gamma$ ,  $t \in [0, T]$ ,  $k \in \mathbb{N}$  it holds that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \mathbb{E}[f_n(X_{t \wedge \tau_k} - b) - f_n(X_0 - b)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( f_n'(X_s - b) \mu(X_s) + \frac{1}{2} f_n''(X_s - b) (\sigma \cdot \sigma)(X_s) \right) ds \right] \\ &= \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \lim_{n \rightarrow \infty} \left( f_n'(X_s - b) \mu(X_s) + \frac{1}{2} f_n''(X_s - b) (\sigma \cdot \sigma)(X_s) \right) ds \right] = \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \mathbb{1}_{\{b\}}(X_s) \mu(b) ds \right]. \end{aligned} \quad (104)$$

Next we obtain from  $\int_0^T |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s. and from the assumption that  $X$  has continuous sample paths that  $\lim_{k \rightarrow \infty} \tau_k = T$   $\mathbb{P}$ -a.s. This, (104), the fact that for all  $b \in \Gamma$  it holds that  $|\mu(b)| > 0$  and Fatou's lemma show that for all  $b \in \Gamma$ ,  $t \in [0, T]$  it holds that

$$\mathbb{E} \left[ \int_0^t \mathbb{1}_{\{X_s=b\}} ds \right] = \mathbb{E} \left[ \int_0^t \mathbb{1}_{\{b\}}(X_s) ds \right] \leq \liminf_{k \rightarrow \infty} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \mathbb{1}_{\{b\}}(X_s) ds \right] = 0. \quad (105)$$

This finishes the proof of Lemma 3.1.  $\square$

**Lemma 3.2.** *Let  $I \subseteq \mathbb{R}$  be an open interval, let  $T \in [0, \infty)$ ,  $x_0 \in \bar{I}$ ,  $J \subseteq \bar{I}$ ,  $\mu \in C(J, \mathbb{R})$ ,  $\sigma \in C(J, [0, \infty))$  satisfy  $I \subseteq J$ ,  $\sigma(I) \subseteq (0, \infty)$ ,  $\sigma \cdot \sigma \in C^1(J, [0, \infty))$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow J$  be an adapted stochastic process with continuous sample paths satisfying*

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad (106)$$



$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , assume that  $|\int_{x_0}^y \frac{1}{\sigma(z)} dz| < \infty$  for all  $y \in J$ , let  $\phi: J \rightarrow \mathbb{R}$  be a function defined by  $\phi(y) := \int_{x_0}^y \frac{1}{\sigma(z)} dz$  for all  $y \in J$  and assume that for all  $b \in J \cap \partial I$  it holds that  $\sigma(b) = 0$  and

$$(\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)) (\mathbb{1}_{\{(\sigma \cdot \sigma)'(b) \geq 0\}} - \mathbb{1}_{\{(\sigma \cdot \sigma)'(b) < 0\}}) > 0, \quad \limsup_{I \ni x \rightarrow b} \frac{|\mu(x) - \mu(b)| + |(\sigma \cdot \sigma)'(x) - (\sigma \cdot \sigma)'(b)|}{\sigma(x)} < \infty. \quad (107)$$

Then it holds that  $\phi$  is injective and absolutely continuous, it holds that

$$\int_0^T \frac{1 + |\mu(X_s)| + |(\sigma \cdot \sigma)'(X_s)|}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds < \infty \quad (108)$$

$\mathbb{P}$ -a.s. and it holds that the mapping  $Y: [0, T] \times \Omega \rightarrow \phi(J)$  defined by  $Y_t := \phi(X_t)$  for all  $t \in [0, T]$  is an adapted stochastic process with continuous sample paths which satisfies

$$Y_t = Y_0 + \int_0^t \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(Y_s)) \mathbb{1}_{\{Y_s \in \phi(I)\}} ds + W_t \quad (109)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ .

*Proof of Lemma 3.2.* First of all, note that the assumption that for all  $b \in J$  it holds that  $|\int_{x_0}^b \frac{1}{\sigma(z)} dz| < \infty$  ensures that  $\phi$  is well-defined and absolutely continuous. Next observe that for all  $x, y \in J$  it holds that  $\int_y^x \frac{1}{\sigma(z)} dz = 0$  if and only if  $\phi(x) = \phi(y)$ . This together with the assumption that  $\sigma \geq 0$  implies that for all  $x, y \in J$  it holds that  $\phi(x) = \phi(y)$  if and only if  $x = y$ , which proves the injectivity of  $\phi$ . In the next step we define families  $\sigma_\varepsilon: J \rightarrow [0, \infty)$ ,  $\varepsilon \in (0, 1)$ , and  $\phi_\varepsilon: J \rightarrow \mathbb{R}$ ,  $\varepsilon \in (0, 1)$ , of functions by

$$\sigma_\varepsilon(z) := \sqrt{\varepsilon + (\sigma \cdot \sigma)(z)} \quad \text{and} \quad \phi_\varepsilon(y) := \int_{x_0}^y \frac{1}{\sigma_\varepsilon(z)} dz \quad (110)$$

for all  $y \in J$ ,  $\varepsilon \in (0, 1)$ . Note that for all  $\varepsilon \in (0, 1)$  it holds that  $\sigma_\varepsilon \in C^1(J, [0, \infty))$  and  $\phi_\varepsilon \in C^2(J, \mathbb{R})$ . Itô's lemma and the fact that for all  $x \in J$ ,  $\varepsilon \in (0, 1)$  it holds that  $(\sigma_\varepsilon \cdot \sigma_\varepsilon)'(x) = (\sigma \cdot \sigma)'(x)$  and  $(\sigma_\varepsilon)'(x) = \frac{(\sigma \cdot \sigma)'(x)}{2\sigma_\varepsilon(x)}$  hence implies that for all  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$  it holds that

$$\begin{aligned} \phi_\varepsilon(X_t) &= \phi_\varepsilon(X_0) + \int_0^t \frac{\mu(X_s)}{\sigma_\varepsilon(X_s)} - \frac{1}{2} \frac{\sigma_\varepsilon'(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)} (\sigma \cdot \sigma)(X_s) ds + \int_0^t \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s \\ &= \phi_\varepsilon(X_0) + \int_0^t \frac{\mu(X_s) - \frac{1}{4}(\sigma \cdot \sigma)'(X_s) \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)}}{\sigma_\varepsilon(X_s)} ds + \int_0^t \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s. \end{aligned} \quad (111)$$

Lemma 3.1 together with the fact that for all  $b \in J \cap \partial I$  it holds that  $|\mu(b)| > 0 = \sigma(b)$  shows that for Lebesgue almost all  $t \in [0, T]$  it holds that

$$\mathbb{P}[X_t \in I] = 1. \quad (112)$$

Next we fix  $y_0 \in I$ , we define a sequence  $\tau_k: \Omega \rightarrow [0, T]$ ,  $k \in \mathbb{N}$ , of stopping times by  $\tau_0 := 0$  and

$$\tau_k := \inf \left( \{T\} \cup \{t \in (\tau_{k-1}, T]: X_t = y_0 \text{ and } \exists s \in [\tau_{k-1}, t]: X_s \in \partial I\} \right) \quad (113)$$

for all  $k \in \mathbb{N}$  and we define a mapping  $\kappa: \Omega \rightarrow \mathbb{N}_0 \cup \{\infty\}$  by  $\kappa := \min(\{k \in \mathbb{N}_0: \tau_k = T\} \cup \{\infty\})$ . Pathwise continuity of  $X$  implies that for all  $\omega \in \Omega$  it holds that  $\kappa(\omega) < \infty$ . Now assumption (107) and the assumptions that  $\sigma \cdot \sigma \in C^1(J, [0, \infty))$ ,  $\sigma(I) \subseteq (0, \infty)$  and  $\mu \in C(J, \mathbb{R})$  imply that for every  $b \in J \cap \partial I$  and every compact set  $K \subseteq I \cup \{b\}$  it holds that

$$\sup_{\varepsilon \in (0, 1)} \sup_{z \in K} \left| \frac{\mu(z) - \mu(b) - \frac{1}{4}((\sigma \cdot \sigma)'(z) - (\sigma \cdot \sigma)'(b)) \frac{(\sigma \cdot \sigma)(z)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(z)}}{\sigma_\varepsilon(z)} \right| \leq \sup_{z \in K \setminus \{b\}} \frac{|\mu(z) - \mu(b)| + |(\sigma \cdot \sigma)'(z) - (\sigma \cdot \sigma)'(b)|}{\sigma(z)} < \infty. \quad (114)$$

This proves that for all  $b \in \partial I \cap J$ ,  $k \in \mathbb{N}$  it holds  $\mathbb{P}$ -a.s. that

$$\begin{aligned} &\mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \int_{\tau_{k-1}}^{\tau_k} \frac{|\mu(X_s) - \mu(b)| + \frac{1}{4}|(\sigma \cdot \sigma)'(X_s) - (\sigma \cdot \sigma)'(b)|}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \\ &\leq \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \int_{\tau_{k-1}}^{\tau_k} \sup \left\{ \frac{|\mu(z) - \mu(b)| + |(\sigma \cdot \sigma)'(z) - (\sigma \cdot \sigma)'(b)|}{\sigma(z)} : z \in [\cup_{u \in [\tau_{k-1}, \tau_k]} \{X_u\}] \setminus \{b\} \right\} ds < \infty. \end{aligned} \quad (115)$$

Next, we define a function  $\text{sgn}: \mathbb{R} \rightarrow \{-1, 1\}$  by  $\text{sgn}(x) := \mathbb{1}_{\{x \geq 0\}} - \mathbb{1}_{\{x < 0\}}$  for all  $x \in \mathbb{R}$ . Then observe that the fact that for all  $b \in J \cap \partial I$  it holds that  $(\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)) \text{sgn}((\sigma \cdot \sigma)'(b)) > 0$  implies that for all  $b \in J \cap \partial I$  it holds that  $\text{sgn}((\sigma \cdot \sigma)'(b))\mu(b) = |\mu(b)|$ . This, (111) and (112) imply that

$$\begin{aligned} &\text{sgn}((\sigma \cdot \sigma)'(b)) \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)}{\sigma_\varepsilon(X_s)} \left( \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)} \right) ds \leq \text{sgn}((\sigma \cdot \sigma)'(b)) \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b) \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)}}{\sigma_\varepsilon(X_s)} ds \\ &= \text{sgn}((\sigma \cdot \sigma)'(b)) \left( \phi_\varepsilon(X_{\tau_k}) - \phi_\varepsilon(X_{\tau_{k-1}}) - \int_{\tau_{k-1}}^{\tau_k} \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s \right. \\ &\quad \left. - \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(X_s) - \mu(b) - \frac{1}{4}((\sigma \cdot \sigma)'(X_s) - (\sigma \cdot \sigma)'(b)) \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)}}{\sigma_\varepsilon(X_s)} ds \right) \\ &= \text{sgn}((\sigma \cdot \sigma)'(b)) \left( \phi_\varepsilon(X_{\tau_k}) - \phi_\varepsilon(X_{\tau_{k-1}}) - \int_{\tau_{k-1}}^{\tau_k} \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s \right. \\ &\quad \left. - \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(X_s) - \mu(b) - \frac{1}{4}((\sigma \cdot \sigma)'(X_s) - (\sigma \cdot \sigma)'(b)) \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_\varepsilon \cdot \sigma_\varepsilon)(X_s)}}{\sigma_\varepsilon(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \right) \end{aligned} \quad (116)$$

$\mathbb{P}$ -a.s. for all  $b \in \partial I \cap J$ ,  $\varepsilon \in (0, 1)$ ,  $k \in \mathbb{N}$ . The monotone convergence theorem shows that for all  $t \in [0, T]$  it holds that  $\lim_{(0,1) \ni \varepsilon \rightarrow 0} \phi_\varepsilon(X_t) = \phi(X_t)$ . Moreover, Doob's martingale inequality, the dominated convergence theorem, (112) and  $\sigma^{-1}(\{0\}) = J \cap \partial I$  imply that

$$\begin{aligned} & \lim_{(0,1) \ni \varepsilon \rightarrow 0} \left\| \sup_{t \in [0, T]} \left| W_t - \int_0^t \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s \right| \right\|_{L^2(\Omega; \mathbb{R})}^2 = \lim_{(0,1) \ni \varepsilon \rightarrow 0} \left\| \sup_{t \in [0, T]} \left| \int_0^t 1 - \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} dW_s \right| \right\|_{L^2(\Omega; \mathbb{R})}^2 \\ & \leq 2 \lim_{(0,1) \ni \varepsilon \rightarrow 0} \mathbb{E} \left[ \int_0^T \left( 1 - \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} \right)^2 ds \right] = 2 \mathbb{E} \left[ \int_0^T \lim_{(0,1) \ni \varepsilon \rightarrow 0} \left( 1 - \frac{\sigma(X_s)}{\sigma_\varepsilon(X_s)} \right)^2 ds \right] \\ & = 2 \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{\sigma(X_s)=0\}} ds \right] = 2 \int_0^T \mathbb{P}[\sigma(X_s) = 0] ds = 2 \int_0^T \mathbb{P}[X_s \in J \cap \partial I] ds = 0. \end{aligned} \quad (117)$$

Consequently, there exists a strictly decreasing sequence  $\varepsilon_n \in (0, \infty)$ ,  $n \in \mathbb{N}$ , such that it holds  $\mathbb{P}$ -a.s. that  $\lim_{N \ni n \rightarrow \infty} \sup_{t \in [0, T]} |W_t - \int_0^t \frac{\sigma(X_s)}{\sigma_{\varepsilon_n}(X_s)} dW_s| = 0$ . This, the monotone convergence theorem, (116) and the dominated convergence theorem together with (115) prove that

$$\begin{aligned} & \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \int_{\tau_{k-1}}^{\tau_k} \frac{|\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)|}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \\ & = \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \lim_{N \ni n \rightarrow \infty} \operatorname{sgn}((\sigma \cdot \sigma)'(b)) \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)}{\sigma_{\varepsilon_n}(X_s)} \left( \frac{(\sigma \cdot \sigma)(X_s)}{(\sigma_{\varepsilon_n} \cdot \sigma_{\varepsilon_n})(X_s)} \right) ds \\ & \leq \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \lim_{N \ni n \rightarrow \infty} \operatorname{sgn}((\sigma \cdot \sigma)'(b)) \left( \phi_{\varepsilon_n}(X_{\tau_k}) - \phi_{\varepsilon_n}(X_{\tau_{k-1}}) - \int_{\tau_{k-1}}^{\tau_k} \frac{\sigma(X_s)}{\sigma_{\varepsilon_n}(X_s)} dW_s \right. \\ & \quad \left. - \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(X_s) - \mu(b) - \frac{1}{4}((\sigma \cdot \sigma)'(X_s) - (\sigma \cdot \sigma)'(b))}{\sigma_{\varepsilon_n}(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \right) \\ & = \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \operatorname{sgn}((\sigma \cdot \sigma)'(b)) \left( \phi(X_{\tau_k}) - \phi(X_{\tau_{k-1}}) - (W_{\tau_k} - W_{\tau_{k-1}}) \right. \\ & \quad \left. - \int_{\tau_{k-1}}^{\tau_k} \frac{\mu(X_s) - \mu(b) - \frac{1}{4}((\sigma \cdot \sigma)'(X_s) - (\sigma \cdot \sigma)'(b))}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \right) < \infty \end{aligned} \quad (118)$$

$\mathbb{P}$ -a.s. for all  $b \in \partial I \cap J$ ,  $k \in \mathbb{N}$ . This and the fact that for all  $b \in \partial I \cap J$  it holds that  $|\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)| > 0$  imply that for all  $k \in \mathbb{N}$  it holds  $\mathbb{P}$ -a.s. that

$$\begin{aligned} & \int_{\tau_{k-1}}^{\tau_k} \frac{1}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \leq \mathbb{1}_{\{\forall u \in [\tau_{k-1}, \tau_k]: X_u \in I\}} \int_{\tau_{k-1}}^{\tau_k} \frac{1}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \\ & + \sum_{b \in \partial I \cap J} \frac{1}{|\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)|} \left[ \mathbb{1}_{\{\exists u \in [\tau_{k-1}, \tau_k]: X_u = b\}} \int_{\tau_{k-1}}^{\tau_k} \frac{|\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)|}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds \right] < \infty. \end{aligned} \quad (119)$$

The fact that for all  $\omega \in \Omega$  it holds that  $\kappa(\omega) < \infty$  hence proves that

$$\int_0^T \frac{1}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds < \infty \quad (120)$$

$\mathbb{P}$ -a.s. This and the fact that  $\mathbb{P}[\sup\{|\mu(X_s)| + |(\sigma \cdot \sigma)'(X_s)|: s \in [0, T]\} < \infty] = 1$  imply (108). Finally, the dominated convergence theorem together with (108), (112) and (111) shows that for every  $t \in [0, T]$  it holds  $\mathbb{P}$ -a.s. that

$$\begin{aligned} & Y_t - Y_0 - \int_0^t \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(Y_s)) \mathbb{1}_{\{Y_s \in \phi(I)\}} ds - W_t \\ & = \phi(X_t) - \phi(X_0) - \int_0^t \frac{\mu(X_s) - \frac{1}{4}(\sigma \cdot \sigma)'(X_s)}{\sigma(X_s)} \mathbb{1}_{\{X_s \in I\}} ds - W_t \\ & = \lim_{n \rightarrow \infty} \left( \phi_{\varepsilon_n}(X_t) - \phi_{\varepsilon_n}(X_0) - \int_0^t \frac{\mu(X_s) - \frac{1}{4}(\sigma_{\varepsilon_n} \cdot \sigma_{\varepsilon_n})'(X_s)}{\sigma_{\varepsilon_n}(X_s)} \mathbb{1}_{\{X_s \in I\}} ds - \int_0^t \frac{\sigma(X_s)}{\sigma_{\varepsilon_n}(X_s)} dW_s \right) \\ & = \lim_{n \rightarrow \infty} \left( \phi_{\varepsilon_n}(X_t) - \phi_{\varepsilon_n}(X_0) - \int_0^t \frac{\mu(X_s) - \frac{1}{4}(\sigma_{\varepsilon_n} \cdot \sigma_{\varepsilon_n})'(X_s)}{\sigma_{\varepsilon_n}(X_s)} ds - \int_0^t \frac{\sigma(X_s)}{\sigma_{\varepsilon_n}(X_s)} dW_s \right) = 0. \end{aligned} \quad (121)$$

This finishes the proof of Lemma 3.2.  $\square$

**Remark 3.3.** In the setting of Lemma 3.2 it holds that if  $(\mu(b) - \frac{1}{4}(\sigma \cdot \sigma)'(b)) (\mathbb{1}_{\{(\sigma \cdot \sigma)'(b) \geq 0\}} - \mathbb{1}_{\{(\sigma \cdot \sigma)'(b) < 0\}}) \leq 0$  for some  $b \in \partial I$ , then the process  $Y$  may not be a solution process of an Itô SDE; see Section XI.1 in Revuz & Yor [32] for the example of the one-dimensional Bessel process.

### 3.2 Properties of drift-implicit Euler approximations for Bessel-type processes

**Lemma 3.4.** Let  $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$  be an  $\mathbb{R}$ -Hilbert space, let  $I \subseteq H$  be a non-empty set, let  $T, L \in [0, \infty)$ , let  $\mu: I \rightarrow H$  be a function satisfying

$$\langle x - y, \mu(x) - \mu(y) \rangle_H \leq L \|x - y\|_H^2 \quad (122)$$

for all  $x, y \in I$ , let  $F: (0, \frac{1}{L}) \times I \rightarrow H$  be a mapping defined by  $F_t(x) := x - t\mu(x)$  for all  $x \in I, t \in (0, \frac{1}{L})$ . Then for every  $t \in (0, \frac{1}{L})$  the function  $F_t: I \rightarrow H$  is injective and for all  $t \in (0, \frac{1}{L})$ ,  $x, y \in F_t(I)$  it holds that

$$\|F_t^{-1}(x) - F_t^{-1}(y)\|_H \leq \frac{1}{(1-tL)} \|x - y\|_H. \quad (123)$$

*Proof of Lemma 3.4.* Observe that the Cauchy-Schwarz inequality and (122) imply that for all  $x, y \in I, t \in (0, \frac{1}{L})$  it holds that

$$\begin{aligned} \|x - y\|_H \|F_t(x) - F_t(y)\|_H &\geq \langle x - y, F_t(x) - F_t(y) \rangle_H = \|x - y\|_H^2 - t \langle x - y, \mu(x) - \mu(y) \rangle_H \\ &\geq \|x - y\|_H^2 - tL \|x - y\|_H^2 = (1 - tL) \|x - y\|_H^2. \end{aligned} \quad (124)$$

This ensures that for every  $t \in (0, \frac{1}{L})$  it holds that the mapping  $F_t$  is injective. In addition, (124) shows that for all  $t \in (0, \frac{1}{L})$ ,  $x, y \in F_t(I)$  it holds that

$$\|x - y\|_H \geq (1 - tL) \|F_t^{-1}(x) - F_t^{-1}(y)\|_H. \quad (125)$$

Estimate (125) implies (123). The proof of Lemma 3.4 is thus completed.  $\square$

**Lemma 3.5.** Let  $I \subseteq \mathbb{R}$  be an open interval, let  $T, L \in [0, \infty)$ ,  $n \in \mathbb{N}_0$ ,  $\theta = (t_0, \dots, t_n) \in [0, T]^{n+1}$  satisfy  $0 = t_0 < t_1 < \dots < t_n = T$  and  $L|\theta| < 1$ , let  $\mu \in \mathcal{L}^0(\bar{I}; \mathbb{R})$  satisfy  $(x - y)(\mu(x) - \mu(y)) \leq L(x - y)^2$  for all  $x, y \in I$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow \bar{I}$  be an adapted stochastic process with continuous sample paths satisfying  $\int_0^t |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s. and

$$X_t = X_0 + \int_0^t \mu(X_s) ds + W_t \quad (126)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , let  $Y: \{t_0, t_1, \dots, t_n\} \times \Omega \rightarrow I$  be a measurable mapping satisfying

$$Y_{t_{k+1}} = Y_{t_k} + \mu(Y_{t_{k+1}})(t_{k+1} - t_k) + W_{t_{k+1}} - W_{t_k} \quad (127)$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n-1\}$  and assume that  $\sup_{i \in \{0, 1, \dots, n\}} \mathbb{P}[X_{t_i} \in \partial I] = 0$ . Then

$$\begin{aligned} (X_{t_k} - Y_{t_k}) &\prod_{i=0}^{k-1} \left(1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}}\right) \\ &= X_0 - Y_0 + \int_0^{t_k} (\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})) \prod_{i \in \{0, \dots, n-1\}: t_{i+1} \leq \lceil s \rceil_\theta} \left(1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}}\right) ds \end{aligned} \quad (128)$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n\}$  and

$$|X_{t_k} - Y_{t_k}| \leq \left(\frac{1}{1-|\theta|L}\right)^k \left(|X_0 - Y_0| + \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| ds\right) \quad (129)$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n\}$ .

*Proof of Lemma 3.5.* First of all, observe that (126) and (127) imply that

$$\begin{aligned} (X_{t_{k+1}} - Y_{t_{k+1}}) &\left(1 - (t_{k+1} - t_k) \frac{\mu(X_{t_{k+1}}) - \mu(Y_{t_{k+1}})}{X_{t_{k+1}} - Y_{t_{k+1}}}\right) \\ &= X_{t_{k+1}} - Y_{t_{k+1}} - (t_{k+1} - t_k) (\mu(X_{t_{k+1}}) - \mu(Y_{t_{k+1}})) \\ &= X_{t_k} - Y_{t_k} + \int_{t_k}^{t_{k+1}} \mu(X_s) ds - \mu(X_{t_{k+1}})(t_{k+1} - t_k) = X_{t_k} - Y_{t_k} + \int_{t_k}^{t_{k+1}} \mu(X_s) - \mu(X_{\lceil s \rceil_\theta}) ds \end{aligned} \quad (130)$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n\}$ . Next we claim that

$$\begin{aligned} (X_{t_j} - Y_{t_j}) &\prod_{i=0}^{j-1} \left(1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}}\right) \\ &= X_0 - Y_0 + \int_0^{t_j} (\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})) \prod_{i \in \{0, 1, \dots, n-1\}: t_{i+1} \leq \lceil s \rceil_\theta} \left(1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}}\right) ds \end{aligned} \quad (131)$$

$\mathbb{P}$ -a.s. for all  $j \in \{1, \dots, n\}$ . We now prove (131) by induction on  $j \in \{1, 2, \dots, n\}$ . The base case  $j = 1$  is (130) with  $k = 0$ . For the induction step we assume that (131) holds for some  $j \in \{1, 2, \dots, n-1\}$ . Then (130) shows that

$$\begin{aligned}
& (X_{t_{j+1}} - Y_{t_{j+1}}) \prod_{i=0}^j \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \\
&= \left( X_{t_j} - Y_{t_j} + \int_{t_j}^{t_{j+1}} \mu(X_s) - \mu(X_{\lceil s \rceil_\theta}) ds \right) \prod_{i=0}^{j-1} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \\
&= \left( \int_{t_j}^{t_{j+1}} \mu(X_s) - \mu(X_{\lceil s \rceil_\theta}) ds \right) \prod_{i=0}^{j-1} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \\
&\quad + X_0 - Y_0 + \int_0^{t_j} (\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})) \prod_{i \in \{0, \dots, n-1\}: t_{i+1} \leq \lceil s \rceil_\theta} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) ds \\
&= X_0 - Y_0 + \int_0^{t_{j+1}} (\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})) \prod_{i \in \{0, \dots, n-1\}: t_{i+1} \leq \lceil s \rceil_\theta} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) ds
\end{aligned} \tag{132}$$

$\mathbb{P}$ -a.s. Induction hence shows (131). Identity (131) proves (128) and it thus remains to prove (129). For this note that (128) and the estimate that  $(t_{k+1} - t_k) \frac{\mu(x) - \mu(y)}{x - y} \leq (t_{k+1} - t_k) L \leq |\theta| L < 1$  for all  $k \in \{0, 1, \dots, n-1\}$ ,  $x, y \in I$  together with the assumption that  $\sup_{i \in \{0, 1, \dots, n\}} \mathbb{P}[X_{t_i} \in \partial I] = 0$  yield that

$$\begin{aligned}
|X_{t_k} - Y_{t_k}| &\leq \left[ \prod_{i=0}^{k-1} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \right]^{-1} \\
&\quad \cdot \left( |X_0 - Y_0| + \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| \prod_{i \in \{0, 1, \dots, n-1\}: t_{i+1} \leq \lceil s \rceil_\theta} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) ds \right) \\
&= \left[ \prod_{i=0}^{k-1} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \right]^{-1} |X_0 - Y_0| \\
&\quad + \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| \left[ \prod_{i \in \{0, 1, \dots, k-1\}: t_{i+1} > \lceil s \rceil_\theta} \left( 1 - (t_{i+1} - t_i) \frac{\mu(X_{t_{i+1}}) - \mu(Y_{t_{i+1}})}{X_{t_{i+1}} - Y_{t_{i+1}}} \right) \right]^{-1} ds \\
&\leq \left[ \prod_{i=0}^{k-1} (1 - |\theta|L) \right]^{-1} |X_0 - Y_0| + \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| \left[ \prod_{i \in \{0, \dots, k-1\}: t_{i+1} > \lceil s \rceil_\theta} (1 - |\theta|L) \right]^{-1} ds \\
&\leq \left( \frac{1}{1 - |\theta|L} \right)^k \left( |X_0 - Y_0| + \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| ds \right)
\end{aligned} \tag{133}$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n\}$ . This finishes the proof of Lemma 3.5.  $\square$

The proof of the following lemma is analogous to the proof of Corollary 2.27 in [24].

**Lemma 3.6.** *Let  $d, m \in \mathbb{N}$ ,  $c, \tilde{c}, T \in [0, \infty)$ ,  $p \in [2, \infty)$ ,  $O \in \mathcal{B}(\mathbb{R}^d)$ ,  $\mu \in \mathcal{L}^0(O; \mathbb{R}^d)$ ,  $\sigma \in \mathcal{L}^0(O; \mathbb{R}^{d \times m})$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a filtered probability space, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $\xi: \Omega \rightarrow O$  be an  $\mathcal{F}_0/\mathcal{B}(O)$ -measurable function with  $\mathbb{E}[\|\mu(\xi)\|_{\mathbb{R}^d}^p] < \infty$ , assume*

$$\begin{aligned}
\langle x - y, \mu(x) - \mu(y) \rangle_{\mathbb{R}^d} &\leq c \|x - y\|_{\mathbb{R}^d}^2 \\
\langle x, \mu(x) \rangle_{\mathbb{R}^d} + \frac{(p-1)}{2} \|\sigma(x)\|_{HS(\mathbb{R}^m, \mathbb{R}^d)}^2 &\leq \tilde{c} (1 + \|x\|_{\mathbb{R}^d}^2)
\end{aligned} \tag{134}$$

for all  $x, y \in O$  and assume that for all  $t \in (0, 1/c)$  it holds that the mapping  $O \ni x \rightarrow x - t\mu(x) \in \mathbb{R}^d$  is surjective. Then there exists a unique family  $Y^h: (\mathbb{N}_0 \cap [0, T/h]) \times \Omega \rightarrow O$ ,  $h \in (0, T] \cap (0, 1/c)$ , of stochastic processes satisfying  $Y_0^h = \xi$  and

$$Y_n^h = Y_{n-1}^h + \mu(Y_n^h) h + \sigma(Y_{n-1}^h) (W_{nh} - W_{(n-1)h}) \tag{135}$$

for all  $n \in \mathbb{N} \cap [0, T/h]$ ,  $h \in (0, T] \cap (0, 1/c)$  and there exists a real number  $\rho \in (0, \infty)$  such that for all  $q \in [0, p/2]$  it holds that

$$\limsup_{h \searrow 0} \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \mathbb{E} \left[ \left( 1 + \|Y_n^h\|_{\mathbb{R}^d}^2 \right)^q \right] \leq e^{\rho T} \mathbb{E} \left[ \left( 1 + \|\xi\|_{\mathbb{R}^d}^2 \right)^q \right] \tag{136}$$

and that

$$\sup_{h \in (0, T] \cap (0, 1/4c)} \sup_{n \in (\mathbb{N}_0 \cap [0, T/h])} \mathbb{E} \left[ 1 + \|Y_n^h\|_{\mathbb{R}^d}^2 \right]^q \leq e^{(4c+\rho)T} \cdot \mathbb{E} \left[ \left\{ 1 + (\|\xi\|_{\mathbb{R}^d} + T\|\mu(\xi)\|_{\mathbb{R}^d})^2 \right\}^q \right]. \tag{137}$$

*Proof of Lemma 3.6.* Throughout this proof, let  $F: (0, 1/c) \times O \rightarrow \mathbb{R}^d$  be a function defined by  $F_h(x) := x - \mu(x)h$  for all  $(h, x) \in (0, 1/c) \times O$ . Then Lemma 3.4 and the surjectivity assumption imply for all  $h \in (0, 1/c)$  that the function  $F_h$  is bijective. This ensures the unique existence of stochastic processes  $Y^h: (\mathbb{N}_0 \cap [0, T/h]) \times \Omega \rightarrow O$ ,  $h \in (0, T] \cap (0, 1/c)$ , satisfying  $Y_0^h = \xi$  and (135). In the next step, note that

$$\|F_h(Y_{n+1}^h)\|_{\mathbb{R}^d}^2 = \|Y_n^h + \sigma(Y_n^h)(W_{(n+1)h} - W_{nh})\|_{\mathbb{R}^d}^2 \quad (138)$$

for all  $n \in (\mathbb{N}_0 \cap [0, T/h])$  and all  $h \in (0, T] \cap (0, 1/c)$ . Lemma 2.26 in [24] hence implies the existence of a real number  $\rho \in \mathbb{R}$  such that

$$\mathbb{E}\left[\left\{1 + \|F_h(Y_{n+1}^h)\|_{\mathbb{R}^d}^2\right\}^q \mid Y_n^h\right] \leq \exp(\rho h) \cdot \left\{1 + \|F_h(Y_n^h)\|_{\mathbb{R}^d}^2\right\}^q \quad (139)$$

$\mathbb{P}$ -a.s. for all  $n \in (\mathbb{N}_0 \cap [0, T/h])$ ,  $h \in (0, T] \cap (0, 1/4c]$  and all  $q \in [0, \frac{\rho}{2}]$ . Next fix a real number  $q \in [0, \frac{\rho}{2}]$  and we now prove (136) for this  $q \in [0, \frac{\rho}{2}]$ . If  $\mathbb{E}[\|\xi\|_{\mathbb{R}^d}^{2q}] = \infty$ , then (136) is trivial. So we assume  $\mathbb{E}[\|\xi\|_{\mathbb{R}^d}^{2q}] < \infty$  for the rest of this proof. Hence, we obtain that  $\mathbb{E}[\|\xi\|_{\mathbb{R}^d}^{2q} + \|\mu(\xi)\|_{\mathbb{R}^d}^{2q}] < \infty$ . Now, for every  $h \in (0, T] \cap (0, 1/4c]$ , we apply Corollary 2.2 in [24] with the Lyapunov-type function  $V: O \rightarrow [0, \infty)$  given by  $V(x) = \{1 + \|F_h(x)\|_{\mathbb{R}^d}^2\}^q$  for all  $x \in O$ , with the truncation function  $\zeta: [0, \infty) \rightarrow (0, \infty]$  given by  $\zeta(t) = \infty$  for all  $t \in [0, \infty)$  and with the sequence  $t_n \in \mathbb{R}$ ,  $n \in \mathbb{N}_0$ , given by  $t_n = \min(nh, T)$  for all  $n \in \mathbb{N}_0$  to obtain

$$\sup_{n \in (\mathbb{N}_0 \cap [0, T/h])} \mathbb{E}\left[\left\{1 + \|F_h(Y_n^h)\|_{\mathbb{R}^d}^2\right\}^q\right] \leq e^{\rho T} \cdot \mathbb{E}\left[\left\{1 + \|F_h(\xi)\|_{\mathbb{R}^d}^2\right\}^q\right] \quad (140)$$

for all  $h \in (0, T] \cap (0, 1/4c]$ . Lemma 2.25 in [24] and the dominated convergence theorem hence give

$$\begin{aligned} \limsup_{h \searrow 0} \sup_{n \in (\mathbb{N}_0 \cap [0, T/h])} \mathbb{E}\left[\left\{1 + \|Y_n^h\|_{\mathbb{R}^d}^2\right\}^q\right] &\leq \limsup_{h \searrow 0} \left(e^{4ch} \cdot e^{\rho T} \cdot \mathbb{E}\left[\left\{1 + \|F_h(\xi)\|_{\mathbb{R}^d}^2\right\}^q\right]\right) \\ &= e^{\rho T} \cdot \mathbb{E}\left[\lim_{(0, T] \ni h \rightarrow 0} \left\{1 + \|F_h(\xi)\|_{\mathbb{R}^d}^2\right\}^q\right] = e^{\rho T} \cdot \mathbb{E}\left[\left\{1 + \|\xi\|_{\mathbb{R}^d}^2\right\}^q\right] \end{aligned} \quad (141)$$

Furthermore, Lemma 2.25 in [24] and (140) yield

$$\sup_{h \in (0, T] \cap (0, 1/4c]} \sup_{n \in (\mathbb{N}_0 \cap [0, T/h])} \mathbb{E}\left[\left\{1 + \|Y_n^h\|_{\mathbb{R}^d}^2\right\}^q\right] \leq e^{4cT} \cdot e^{\rho T} \cdot \mathbb{E}\left[\left\{1 + (\|\xi\|_{\mathbb{R}^d} + T\|\mu(\xi)\|_{\mathbb{R}^d})^2\right\}^q\right] \quad (142)$$

and this completes the proof of Lemma 3.6.  $\square$

**Lemma 3.7.** Let  $d, m \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$ ,  $T, c \in [0, \infty)$ ,  $\theta = (t_0, \dots, t_n) \in [0, T]^{n+1}$  satisfy  $0 = t_0 < t_1 < \dots < t_n = T$  and  $4c|\theta| \leq 1$ , let  $O \in \mathcal{B}(\mathbb{R}^d)$ ,  $\mu \in \mathcal{L}^0(O; \mathbb{R}^d)$ ,  $\sigma \in \mathcal{L}^0(O; \mathbb{R}^{d \times m})$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a filtered probability space, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $Y: \{t_0, t_1, \dots, t_n\} \times \Omega \rightarrow O$  be a measurable mapping satisfying

$$Y_{t_{k+1}} = Y_{t_k} + \mu(Y_{t_{k+1}})(t_{k+1} - t_k) + \sigma(Y_{t_k})(W_{t_{k+1}} - W_{t_k}) \quad (143)$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, 1, \dots, n-1\}$  and assume that for all  $x \in O$  it holds that  $\langle x, \mu(x) \rangle_{\mathbb{R}^d} \leq c(1 + \|x\|_{\mathbb{R}^d}^2)$ . Then it holds for all  $k \in \{0, \dots, n\}$ ,  $p \in [2, \infty)$  that

$$\begin{aligned} &\left\| \sup_{i \in \{0, 1, \dots, k\}} [1 + \|Y_{t_i}\|_{\mathbb{R}^d}^2] \right\|_{L^p(\Omega; \mathbb{R})} \\ &\leq e^{4ct_k} \left[1 + \|Y_0\|_{L^{2p}(\Omega; \mathbb{R}^d)}^2\right] + \left[ \frac{2p^3}{p-1} \sum_{i=0}^{k-1} e^{8c(t_k - t_i)} \|\sigma(Y_{t_i})^* Y_{t_i}\|_{L^p(\Omega; \mathbb{R}^m)}^2 (t_{i+1} - t_i) \right]^{1/2} \\ &\quad + p(2p-1) \left[ \sum_{i=0}^{k-1} e^{4c(t_k - t_i)} \|\sigma(Y_{t_i})\|_{L^{2p}(\Omega; HS(\mathbb{R}^m, \mathbb{R}^d))}^2 (t_{i+1} - t_i) \right]. \end{aligned} \quad (144)$$

*Proof of Lemma 3.7.* First of all, Lemma 2.25 in [24] together with  $4c|\theta| \leq 1$  implies that for all  $k \in \{1, \dots, n\}$  it holds that

$$1 + \|Y_{t_k}\|_{\mathbb{R}^d}^2 \leq e^{4c(t_k - t_{k-1})} [1 + \|Y_{t_k} - \mu(Y_{t_k})(t_k - t_{k-1})\|_{\mathbb{R}^d}^2]. \quad (145)$$

This together with (143) yields that for all  $k \in \{1, \dots, n\}$  it holds that

$$\begin{aligned} 1 + \|Y_{t_k}\|_{\mathbb{R}^d}^2 &\leq e^{4c(t_k - t_{k-1})} \left[1 + \|Y_{t_{k-1}} + \sigma(Y_{t_{k-1}})(W_{t_k} - W_{t_{k-1}})\|_{\mathbb{R}^d}^2\right] \\ &= e^{4c(t_k - t_{k-1})} \left[1 + \|Y_{t_{k-1}}\|_{\mathbb{R}^d}^2 + 2 \int_{t_{k-1}}^{t_k} (Y_{t_{k-1}})^* \sigma(Y_{t_{k-1}}) dW_s + \|\sigma(Y_{t_{k-1}})(W_{t_k} - W_{t_{k-1}})\|_{\mathbb{R}^d}^2\right] \end{aligned} \quad (146)$$

$\mathbb{P}$ -a.s. A straight forward induction on  $k \in \{1, \dots, n-1\}$  then shows that for all  $k \in \{0, \dots, n\}$  it holds that

$$\begin{aligned} 1 + \|Y_{t_k}\|_{\mathbb{R}^d}^2 &\leq e^{4ct_k} [1 + \|Y_0\|_{\mathbb{R}^d}^2] + 2 \int_{t_0}^{t_k} e^{4c(t_k - \lfloor s \rfloor_\theta)} (Y_{\lfloor s \rfloor_\theta})^* \sigma(Y_{\lfloor s \rfloor_\theta}) dW_s \\ &\quad + \sum_{i=0}^{k-1} e^{4c(t_k - t_i)} \|\sigma(Y_{t_i})(W_{t_{i+1}} - W_{t_i})\|_{\mathbb{R}^d}^2 \end{aligned} \quad (147)$$

$\mathbb{P}$ -a.s. Estimate (147) and the Burkholder-Davis-Gundy-type inequalities in Lemma 7.2 and Lemma 7.7 in Da Prato & Zabczyk [9] ensure that for all  $k \in \{0, \dots, n\}$ ,  $p \in [2, \infty)$  it holds that

$$\begin{aligned}
& \left\| \sup_{i \in \{0, 1, \dots, k\}} [1 + \|Y_{t_i}\|_{\mathbb{R}^d}^2] \right\|_{L^p(\Omega; \mathbb{R})} \\
& \leq e^{4ct_k} \|1 + \|Y_0\|_{\mathbb{R}^d}^2\|_{L^p(\Omega; \mathbb{R})} + 2 \left\| \sup_{s \in [t_0, t_k]} \left| \int_{t_0}^s e^{4c(t_k - \lfloor s \rfloor_\theta)} (Y_{\lfloor s \rfloor_\theta})^* \sigma(Y_{\lfloor s \rfloor_\theta}) dW_s \right| \right\|_{L^p(\Omega; \mathbb{R})} \\
& \quad + \left\| \sum_{i=0}^{k-1} e^{4c(t_k - t_i)} \|\sigma(Y_{t_i})(W_{t_{i+1}} - W_{t_i})\|_{\mathbb{R}^d}^2 \right\|_{L^p(\Omega; \mathbb{R})} \\
& \leq e^{4ct_k} [1 + \|Y_0\|_{L^{2p}(\Omega; \mathbb{R}^d)}^2] + 2 \sqrt{\frac{p^3}{2(p-1)}} \int_{t_0}^{t_k} e^{8c(t_k - \lfloor s \rfloor_\theta)} \|(Y_{\lfloor s \rfloor_\theta})^* \sigma(Y_{\lfloor s \rfloor_\theta})\|_{L^p(\Omega; HS(\mathbb{R}^m, \mathbb{R}))}^2 ds \\
& \quad + \sum_{i=0}^{k-1} e^{4c(t_k - t_i)} \|\sigma(Y_{t_i})(W_{t_{i+1}} - W_{t_i})\|_{L^{2p}(\Omega; \mathbb{R}^d)}^2 \\
& \leq e^{4ct_k} [1 + \|Y_0\|_{L^{2p}(\Omega; \mathbb{R}^d)}^2] + \left[ \frac{2p^3}{p-1} \sum_{i=0}^{k-1} e^{8c(t_k - t_i)} \|\sigma(Y_{t_i})^* Y_{t_i}\|_{L^p(\Omega; \mathbb{R}^m)}^2 (t_{i+1} - t_i) \right]^{1/2} \\
& \quad + p(2p-1) \left[ \sum_{i=0}^{k-1} e^{4c(t_k - t_i)} \|\sigma(Y_{t_i})\|_{L^{2p}(\Omega; HS(\mathbb{R}^m, \mathbb{R}^d))}^2 (t_{i+1} - t_i) \right].
\end{aligned} \tag{148}$$

This finishes the proof of Lemma 3.7.  $\square$

The next result, Corollary 3.8, proves, under suitable assumptions, uniform moment bounds (see (151) below) for a family of fully-drift implicit Euler approximations. Corollary 3.8 follows immediately from a combination of Lemma 3.7 and Lemma 3.6.

**Corollary 3.8.** *Let  $d, m \in \mathbb{N}$ ,  $c, \bar{c}, \tilde{c}, T \in [0, \infty)$ ,  $p \in [2, \infty)$ ,  $O \in \mathcal{B}(\mathbb{R}^d)$ ,  $\mu \in \mathcal{L}^0(O; \mathbb{R}^d)$ ,  $\sigma \in \mathcal{L}^0(O; \mathbb{R}^{d \times m})$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a filtered probability space, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $\xi: \Omega \rightarrow O$  be an  $\mathcal{F}_0/\mathcal{B}(O)$ -measurable function with  $\|\xi\|_{\mathbb{R}^d} + \|\mu(\xi)\|_{\mathbb{R}^d} \|_{L^{(\bar{c}+1 \vee \bar{c})p}(\Omega; \mathbb{R})} < \infty$ , assume*

$$\begin{aligned}
& \|\sigma(x)\|_{HS(\mathbb{R}^m, \mathbb{R}^d)} \leq \tilde{c} (1 + \|x\|_{\mathbb{R}^d}^{\bar{c}}) \\
& \langle x - y, \mu(x) - \mu(y) \rangle_{\mathbb{R}^d} \leq c \|x - y\|_{\mathbb{R}^d}^2 \\
& \langle x, \mu(x) \rangle_{\mathbb{R}^d} + \frac{(\bar{c}+1 \vee \bar{c})p-1}{2} \|\sigma(x)\|_{HS(\mathbb{R}^m, \mathbb{R}^d)}^2 \leq \tilde{c} (1 + \|x\|_{\mathbb{R}^d}^2)
\end{aligned} \tag{149}$$

for all  $x, y \in O$  and assume that for all  $t \in (0, 1/c)$  it holds that the mapping  $O \ni x \rightarrow x - t\mu(x) \in \mathbb{R}^d$  is surjective. Then there exists a unique family  $Y^h: (\mathbb{N}_0 \cap [0, T/h]) \times \Omega \rightarrow O$ ,  $h \in (0, T] \cap (0, 1/c)$ , of stochastic processes satisfying  $Y_0^h = \xi$  and

$$Y_n^h = Y_{n-1}^h + \mu(Y_n^h) h + \sigma(Y_{n-1}^h) (W_{nh} - W_{(n-1)h}) \tag{150}$$

for all  $n \in \mathbb{N} \cap [0, T/h]$ ,  $h \in (0, T] \cap (0, 1/c)$  and it holds that

$$\sup_{h \in (0, T] \cap (0, \frac{1}{4c})} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \|Y_n^h\|_{\mathbb{R}^d} \right\|_{L^p(\Omega; \mathbb{R})} < \infty. \tag{151}$$

### 3.3 Strong convergence rates for drift-implicit Euler approximations of Bessel-type processes

The following corollary establishes strong convergence rates for drift-implicit Euler approximations of Bessel-type processes.

**Corollary 3.9.** *Let  $T, L, c \in [0, \infty)$ ,  $\varepsilon \in (0, \infty)$ ,  $\gamma \in (0, 1]$ ,  $\mu \in \mathcal{L}^0([0, \infty); \mathbb{R})$ ,  $n \in \mathbb{N}_0$ ,  $\theta = (t_0, \dots, t_n) \in [0, T]^{n+1}$  satisfy  $0 = t_0 < t_1 < \dots < t_n = T$ ,  $L|\theta| < 1$  and  $(x - y)(\mu(x) - \mu(y)) \leq L(x - y)^2$  and  $|\mu(x) - \mu(y)| \leq c|x - y|(\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} + x^c + y^c)$  for all  $x, y \in (0, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying  $\int_0^T |\mu(X_s)| ds < \infty$   $\mathbb{P}$ -a.s.,  $\sup_{s \in [0, T]} \mathbb{P}[X_s = 0] = 0$  and*

$$X_t = X_0 + \int_0^t \mu(X_s) ds + W_t \tag{152}$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , let  $Y: \{t_0, t_1, \dots, t_n\} \times \Omega \rightarrow [0, \infty)$  be a measurable mapping satisfying  $Y_0 = X_0$  and

$$Y_{t_{k+1}} = Y_{t_k} + \mu(Y_{t_{k+1}})(t_{k+1} - t_k) + W_{t_{k+1}} - W_{t_k} \tag{153}$$

$\mathbb{P}$ -a.s. for all  $k \in \{0, \dots, n-1\}$ . Then

$$\begin{aligned} \left\| \sup_{k \in \{0, 1, \dots, n\}} |X_{t_k} - Y_{t_k}| \right\|_{L^1(\Omega; \mathbb{R})} &\leq 2Tc \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}| \right\|_{L^{\gamma(1+1/\varepsilon)}(\Omega; \mathbb{R})}^\gamma \\ &\cdot \sup_{u \in [0, T]} \left[ \left\| \frac{2}{(X_u)^{1+\gamma}} \right\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + \frac{\gamma}{1+\gamma} \|(X_u)^{1/\gamma-\gamma}\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + 2\|(X_u)^{c+1-\gamma}\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + \frac{1}{1+\gamma} \right]. \end{aligned} \quad (154)$$

*Proof of Corollary 3.9.* Observe that Young's inequality implies that for all  $x, y \in (0, \infty)$  it holds that

$$\begin{aligned} (x+y)^{(1-\gamma)} \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \right) &= \frac{x+y}{x(x+y)^\gamma} + \frac{x+y}{y(x+y)^\gamma} + \frac{x+y}{xy(x+y)^\gamma} = \frac{2+\frac{x}{y}+\frac{y}{x}}{(x+y)^\gamma} + \frac{\frac{1}{y}+\frac{1}{x}}{(x+y)^\gamma} \\ &\leq \frac{1}{x^\gamma} + \frac{1}{y^\gamma} + \frac{x^{1-\gamma}}{y} + \frac{y^{1-\gamma}}{x} + \frac{1}{y^{1+\gamma}} + \frac{1}{x^{1+\gamma}} \\ &\leq \frac{\gamma}{1+\gamma} \frac{1}{x^{1+\gamma}} + \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \frac{1}{y^{1+\gamma}} + \frac{1}{1+\gamma} + \frac{1}{1+\gamma} \frac{1}{y^{1+\gamma}} + \frac{\gamma}{1+\gamma} x^{\frac{(1-\gamma)(1+\gamma)}{\gamma}} + \frac{1}{1+\gamma} \frac{1}{x^{1+\gamma}} + \frac{\gamma}{1+\gamma} y^{\frac{(1-\gamma)(1+\gamma)}{\gamma}} \\ &\quad + \frac{1}{y^{1+\gamma}} + \frac{1}{x^{1+\gamma}} \\ &= \frac{2}{x^{1+\gamma}} + \frac{2}{y^{1+\gamma}} + \frac{\gamma}{1+\gamma} x^{\frac{1}{\gamma}-\gamma} + \frac{\gamma}{1+\gamma} y^{\frac{1}{\gamma}-\gamma} + \frac{2}{1+\gamma} \end{aligned} \quad (155)$$

and

$$\begin{aligned} (x+y)^{1-\gamma} (x^c + y^c) &= \frac{(x+y)x^c}{(x+y)^\gamma} + \frac{(x+y)y^c}{(x+y)^\gamma} \leq x^{c+1-\gamma} + x^c y^{1-\gamma} + y^{c+1-\gamma} + y^c x^{1-\gamma} \\ &\leq x^{c+1-\gamma} + y^{1-\gamma+c} \frac{1-\gamma}{1-\gamma+c} + x^{1-\gamma+c} \frac{c}{1-\gamma+c} + y^{c+1-\gamma} + x^{1-\gamma+c} \frac{1-\gamma}{1-\gamma+c} + y^{1-\gamma+c} \frac{c}{1-\gamma+c} \\ &= 2x^{c+1-\gamma} + 2y^{c+1-\gamma}. \end{aligned} \quad (156)$$

Moreover, observe that Lemma 3.5, the assumption that  $\sup_{s \in [0, T]} \mathbb{P}[X_s = 0] = 0$  and Hölder's inequality show that

$$\begin{aligned} \left\| \sup_{k \in \{0, 1, \dots, n\}} |X_{t_k} - Y_{t_k}| \right\|_{L^1(\Omega; \mathbb{R})} &\leq \left\| \sup_{k \in \{0, 1, \dots, n\}} \left[ \left( \frac{1}{1-|\theta|L} \right)^k \int_0^{t_k} |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| ds \right] \right\|_{L^1(\Omega; \mathbb{R})} \\ &= \left\| \left( \frac{1}{1-|\theta|L} \right)^n \int_0^T |\mu(X_s) - \mu(X_{\lceil s \rceil_\theta})| \mathbb{1}_{\{X_s, X_{\lceil s \rceil_\theta} \in (0, \infty)\}} ds \right\|_{L^1(\Omega; \mathbb{R})} \\ &\leq c \left( \frac{1}{1-|\theta|L} \right)^n \left\| \int_0^T |X_s - X_{\lceil s \rceil_\theta}| \left( \frac{1}{X_s} + \frac{1}{X_{\lceil s \rceil_\theta}} + \frac{1}{X_s X_{\lceil s \rceil_\theta}} + (X_s)^c + (X_{\lceil s \rceil_\theta})^c \right) ds \right\|_{L^1(\Omega; \mathbb{R})} \\ &\leq c \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}|^\gamma \int_0^T |X_s + X_{\lceil s \rceil_\theta}|^{1-\gamma} \right. \\ &\quad \cdot \left. \left( \frac{1}{X_s} + \frac{1}{X_{\lceil s \rceil_\theta}} + \frac{1}{X_s X_{\lceil s \rceil_\theta}} + (X_s)^c + (X_{\lceil s \rceil_\theta})^c \right) ds \right\|_{L^1(\Omega; \mathbb{R})} \\ &\leq c \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}|^\gamma \right\|_{L^{1+1/\varepsilon}(\Omega; \mathbb{R})} \\ &\quad \cdot \left\| \int_0^T |X_s + X_{\lceil s \rceil_\theta}|^{1-\gamma} \left( \frac{1}{X_s} + \frac{1}{X_{\lceil s \rceil_\theta}} + \frac{1}{X_s X_{\lceil s \rceil_\theta}} + (X_s)^c + (X_{\lceil s \rceil_\theta})^c \right) ds \right\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} \\ &\leq c \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}| \right\|_{L^{\gamma(1+1/\varepsilon)}(\Omega; \mathbb{R})}^\gamma \\ &\quad \cdot \int_0^T \| |X_s + X_{\lceil s \rceil_\theta}|^{1-\gamma} \left( \frac{1}{X_s} + \frac{1}{X_{\lceil s \rceil_\theta}} + \frac{1}{X_s X_{\lceil s \rceil_\theta}} + (X_s)^c + (X_{\lceil s \rceil_\theta})^c \right) \|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} ds. \end{aligned} \quad (157)$$

Inequality (157), inequality (155) and inequality (156) imply that

$$\begin{aligned} \left\| \sup_{k \in \{0, 1, \dots, n\}} |X_{t_k} - Y_{t_k}| \right\|_{L^1(\Omega; \mathbb{R})} &\leq c \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}| \right\|_{L^{\gamma(1+1/\varepsilon)}(\Omega; \mathbb{R})}^\gamma \int_0^T \left\| \frac{2}{(X_{\lceil s \rceil_\theta})^{1+\gamma}} + \frac{2}{(X_s)^{1+\gamma}} \right. \\ &\quad \left. + \frac{\gamma}{1+\gamma} (X_{\lceil s \rceil_\theta})^{\frac{1}{\gamma}-\gamma} + \frac{\gamma}{1+\gamma} (X_s)^{\frac{1}{\gamma}-\gamma} + \frac{2}{1+\gamma} + 2(X_{\lceil s \rceil_\theta})^{c+1-\gamma} + 2(X_s)^{c+1-\gamma} \right\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} ds \\ &\leq 2Tc \left( \frac{1}{1-|\theta|L} \right)^n \left\| \sup_{u \in [0, T]} |X_u - X_{\lceil u \rceil_\theta}| \right\|_{L^{\gamma(1+1/\varepsilon)}(\Omega; \mathbb{R})}^\gamma \\ &\quad \cdot \sup_{u \in [0, T]} \left[ \left\| \frac{2}{(X_u)^{1+\gamma}} \right\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + \frac{\gamma}{1+\gamma} \|(X_u)^{\frac{1}{\gamma}-\gamma}\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + 2\|(X_u)^{c+1-\gamma}\|_{L^{1+\varepsilon}(\Omega; \mathbb{R})} + \frac{1}{1+\gamma} \right]. \end{aligned} \quad (158)$$

This finishes the proof of Corollary 3.9.  $\square$

**Remark 3.10.** In the setting of Corollary 3.9, Hölder's inequality implies that for all  $p \in [1, \infty)$ ,  $\kappa \in [0, \infty]$  it holds that

$$\begin{aligned}
& \left\| \sup_{k \in \{0,1,\dots,n\}} |X_{t_k} - Y_{t_k}| \right\|_{L^p(\Omega; \mathbb{R})} = \left\| \sup_{k \in \{0,1,\dots,n\}} \left[ |X_{t_k} - Y_{t_k}|^{\frac{1}{p(1+\kappa)}} |X_{t_k} - Y_{t_k}|^{1-\frac{1}{p(1+\kappa)}} \right] \right\|_{L^p(\Omega; \mathbb{R})} \\
& \leq \left\| \sup_{k \in \{0,1,\dots,n\}} |X_{t_k} - Y_{t_k}|^{\frac{1}{p(1+\kappa)}} \right\|_{L^{p(1+\kappa)}(\Omega; \mathbb{R})} \left\| \sup_{k \in \{0,1,\dots,n\}} |X_{t_k} - Y_{t_k}|^{1-\frac{1}{p(1+\kappa)}} \right\|_{L^{p(1+\kappa)}(\Omega; \mathbb{R})} \\
& \leq \left\| \sup_{k \in \{0,1,\dots,n\}} |X_{t_k} - Y_{t_k}| \right\|_{L^1(\Omega; \mathbb{R})}^{\frac{1}{p(1+\kappa)}} \left[ \left\| \sup_{k \in \{0,1,\dots,n\}} |X_{t_k}| \right\|_{L^{p+\frac{p-1}{\kappa}}(\Omega; \mathbb{R})}^{1-\frac{1}{p(1+\kappa)}} + \left\| \sup_{k \in \{0,1,\dots,n\}} |Y_{t_k}| \right\|_{L^{p+\frac{p-1}{\kappa}}(\Omega; \mathbb{R})}^{1-\frac{1}{p(1+\kappa)}} \right].
\end{aligned} \tag{159}$$

### 3.4 Strong convergence rates for drift-implicit square root Euler approximations of Cox-Ingersoll-Ross-type processes

In the next two lemmas we present elementary properties of the drift coefficient of the transformed SDE derived in Lemma 3.2.

**Lemma 3.11.** Let  $\mu \in C^1([0, \infty), \mathbb{R})$ ,  $\sigma \in C([0, \infty), [0, \infty))$  satisfy  $\sigma(0) = 0$ ,  $\sigma((0, \infty)) \subseteq (0, \infty)$ ,  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$ ,  $\mu(0) > \frac{(\sigma \cdot \sigma)'(0)}{4} > 0$  and  $\int_1^\infty \frac{1}{\sigma(z)} dz = \infty$ , let  $\phi: [0, \infty) \rightarrow [0, \infty)$  be the function defined by  $\phi(y) := \int_0^y \frac{1}{\sigma(z)} dz$  for all  $y \in [0, \infty)$  and let  $g: (0, \infty) \rightarrow \mathbb{R}$  be the function defined by

$$g(x) := \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(x)) \tag{160}$$

for all  $x \in (0, \infty)$ . Then  $\phi$  is well-defined and bijective,  $g$  is well-defined and continuously differentiable,  $\lim_{(0, \infty) \ni x \rightarrow 0} \frac{\sigma(x)}{\sqrt{x}} = \sqrt{(\sigma \cdot \sigma)'(0)} \in (0, \infty)$ ,

$$\lim_{(0, \infty) \ni x \rightarrow 0} xg(x) = \frac{2\mu(0)}{(\sigma \cdot \sigma)'(0)} - \frac{1}{2} \quad \text{and} \quad \limsup_{(0, \infty) \ni x \rightarrow 0} \left| g'(x) + \frac{1}{x^2} \left( \frac{2\mu(0)}{(\sigma \cdot \sigma)'(0)} - \frac{1}{2} \right) \right| < \infty. \tag{161}$$

*Proof of Lemma 3.11.* First of all, we define  $\alpha := \frac{2\mu(0)}{(\sigma \cdot \sigma)'(0)} - \frac{1}{2} \in (0, \infty)$ . Next note that the assumptions  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$  and  $\sigma((0, \infty)) \subseteq (0, \infty)$  ensure that  $\sigma|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . Furthermore, observe that the assumptions  $\mu \in C^1([0, \infty), \mathbb{R})$ ,  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$  and  $\mu(0) > \frac{(\sigma \cdot \sigma)'(0)}{4} > 0$  imply that there exist real numbers  $\varepsilon \in (0, 1)$  and  $c_1 \in (0, \infty)$ , which we fix for the rest of this proof, such that for all  $x \in (0, \varepsilon)$  it holds that

$$0 < \frac{1}{4}(\sigma \cdot \sigma)'(0) \leq (\sigma \cdot \sigma)'(0) - c_1 x \leq (\sigma \cdot \sigma)'(x) \leq (\sigma \cdot \sigma)'(0) + c_1 x \quad \text{and} \tag{162}$$

$$0 < \frac{\mu(0)}{2} - \frac{(\sigma \cdot \sigma)'(0)}{8} \leq \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} - c_1 x \leq \mu(x) - \frac{(\sigma \cdot \sigma)'(x)}{4} \leq \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} + c_1 x. \tag{163}$$

In the next step we observe that the assumption  $\sigma(0) = 0$  shows that

$$\lim_{(0, \infty) \ni x \rightarrow 0} \frac{\sigma(x)}{\sqrt{x}} = \sqrt{\lim_{(0, \infty) \ni x \rightarrow 0} \frac{(\sigma \cdot \sigma)(x)}{x}} = \sqrt{(\sigma \cdot \sigma)'(0)} \in (0, \infty). \tag{164}$$

Hence, there exist real numbers  $\delta \in (0, 1)$  and  $\lambda \in (0, \infty)$  such that for all  $x \in (0, \delta)$  it holds that  $\sigma(x) \geq \lambda\sqrt{x}$ . This, the continuity of  $\sigma$  and the assumption that  $\sigma((0, \infty)) \subseteq (0, \infty)$  imply that for all  $y \in (0, \infty)$  it holds that

$$\int_0^y \frac{1}{\sigma(z)} dz \leq \int_0^\delta \frac{1}{\sigma(z)} dz + \left| \int_\delta^y \frac{1}{\sigma(z)} dz \right| \leq \int_0^\delta \frac{1}{\lambda\sqrt{z}} dz + \left| \int_\delta^y \frac{1}{\sigma(z)} dz \right| < \infty. \tag{165}$$

This ensures that  $\phi$  is well-defined. Furthermore, observe that  $\phi$  is strictly increasing and continuous and note that  $\phi|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . This and the assumption that  $\int_1^\infty \frac{1}{\sigma(z)} dz = \infty$  show that  $\phi: [0, \infty) \rightarrow [0, \infty)$  is bijective, that  $\phi^{-1}: [0, \infty) \rightarrow [0, \infty)$  is also strictly increasing and continuous and that  $\phi^{-1}|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . In the next step we observe that fact that for all  $x \in (0, \infty)$  it holds that  $\sigma(\phi^{-1}(x)) > 0$  ensures that  $g$  is well-defined. Furthermore, we note that l'Hospital's rule, the fact that  $\phi(0) = 0$  and (164) show that

$$\lim_{(0, \infty) \ni x \rightarrow 0} \frac{\phi(x)}{\sqrt{x}} = \lim_{(0, \infty) \ni x \rightarrow 0} 2\phi'(x)\sqrt{x} = \lim_{(0, \infty) \ni x \rightarrow 0} \frac{2\sqrt{x}}{\sigma(x)} = \frac{2}{\sqrt{(\sigma \cdot \sigma)'(0)}} \in (0, \infty). \tag{166}$$

In addition, observe that (164), (166) and the identity  $\phi^{-1}(0) = \phi(0) = 0$  imply that

$$\lim_{(0, \infty) \ni x \rightarrow 0} \frac{x}{\sigma(\phi^{-1}(x))} = \lim_{(0, \infty) \ni x \rightarrow 0} \frac{\phi(x)}{\sigma(x)} = \lim_{(0, \infty) \ni x \rightarrow 0} \frac{\phi(x)}{\sqrt{x}} \cdot \lim_{(0, \infty) \ni x \rightarrow 0} \frac{\sqrt{x}}{\sigma(x)} = \frac{2}{(\sigma \cdot \sigma)'(0)} \in (0, \infty). \tag{167}$$

This shows that

$$\begin{aligned}
& \lim_{(0, \infty) \ni x \rightarrow 0} [x \cdot g(x)] = \lim_{(0, \infty) \ni x \rightarrow 0} \left[ x \cdot \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(x)) \right] \\
& = \left[ \lim_{(0, \infty) \ni x \rightarrow 0} \left( \frac{x}{\sigma(\phi^{-1}(x))} \right) \right] \left[ \lim_{(0, \infty) \ni x \rightarrow 0} (\mu(x) - \frac{1}{4}(\sigma \cdot \sigma)'(x)) \right] = \frac{2\mu(0)}{(\sigma \cdot \sigma)'(0)} - \frac{1}{2} = \alpha.
\end{aligned} \tag{168}$$



Moreover, observe that identities  $(\sigma \cdot \sigma)'(x) = 2\sigma(x)\sigma'(x)$  and  $(\phi^{-1})'(\phi(x)) = \sigma(x)$  for all  $x \in (0, \infty)$  imply that for all  $y \in (0, \infty)$  it holds that

$$g'(\phi(y)) = \left( \frac{\left( \mu' - \frac{(\sigma \cdot \sigma)''}{4} \right) \sigma - \left( \mu - \frac{(\sigma \cdot \sigma)'}{4} \right) \sigma'}{\sigma \cdot \sigma} \right)(y) \cdot \sigma(y) = \mu'(y) - \frac{(\sigma \cdot \sigma)''(y)}{4} - \left( \frac{\left( \mu - \frac{(\sigma \cdot \sigma)'}{4} \right) (\sigma \cdot \sigma)'}{2(\sigma \cdot \sigma)} \right)(y). \quad (169)$$

In the next step we define a real number  $c_2 := \frac{8c_1}{9}((\sigma \cdot \sigma)'(0))^{-3/2} \in (0, \infty)$ . The assumption that  $\sigma(0) = 0$  and estimate (162) show then that for all  $y \in (0, \varepsilon)$  it holds that

$$\begin{aligned} 0 \leq \phi(y) &= \int_0^y \frac{1}{\sigma(z)} dz = \int_0^y \frac{1}{\sqrt{\int_0^z (\sigma \cdot \sigma)'(x) dx}} dz \leq \int_0^y \frac{1}{\sqrt{(\sigma \cdot \sigma)'(0)z - c_1 z^2}} dz \\ &= \int_0^y \frac{1}{\sqrt{(\sigma \cdot \sigma)'(0)z}} dz + \int_0^y \frac{\sqrt{(\sigma \cdot \sigma)'(0)z} - \sqrt{(\sigma \cdot \sigma)'(0)z - c_1 z^2}}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{(\sigma \cdot \sigma)'(0)z - c_1 z^2}} dz \\ &= \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \int_0^y \frac{c_1 z^2}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{(\sigma \cdot \sigma)'(0)z - c_1 z^2} (\sqrt{(\sigma \cdot \sigma)'(0)z} + \sqrt{(\sigma \cdot \sigma)'(0)z - c_1 z^2})} dz \\ &\leq \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \int_0^y \frac{c_1 z^2}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{\frac{1}{4}(\sigma \cdot \sigma)'(0)z} (\sqrt{(\sigma \cdot \sigma)'(0)z} + \sqrt{\frac{1}{4}(\sigma \cdot \sigma)'(0)z})} dz \\ &= \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \int_0^y \frac{4c_1 z^2}{3((\sigma \cdot \sigma)'(0)z)^{3/2}} dz = \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \frac{8c_1 y^{3/2}}{9((\sigma \cdot \sigma)'(0))^{3/2}} = \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + c_2 y^{3/2}. \end{aligned} \quad (170)$$

Estimate (162), estimate (163) and the identity  $\alpha = \frac{4\mu(0) - (\sigma \cdot \sigma)'(0)}{2(\sigma \cdot \sigma)'(0)}$  therefore imply that for all  $y \in (0, \varepsilon)$  it holds that

$$\begin{aligned} &(\phi(y))^2 \cdot \left( \frac{\left( \mu - \frac{(\sigma \cdot \sigma)'}{4} \right) (\sigma \cdot \sigma)'}{2(\sigma \cdot \sigma)} \right)(y) - \alpha \leq \left( \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} + c_2 y^{3/2} \right)^2 \frac{\left( \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} + c_1 y \right) ((\sigma \cdot \sigma)'(0) + c_1 y)}{2 \int_0^y (\sigma \cdot \sigma)'(x) dx} - \alpha \\ &\leq \left( \frac{4y}{(\sigma \cdot \sigma)'(0)} + \frac{4c_2 y^2}{\sqrt{(\sigma \cdot \sigma)'(0)}} + c_2^2 y^3 \right) \frac{\left( \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} + c_1 y \right) ((\sigma \cdot \sigma)'(0) + c_1 y)}{2(\sigma \cdot \sigma)'(0)y - c_1 y^2} - \frac{4\mu(0) - (\sigma \cdot \sigma)'(0)}{2(\sigma \cdot \sigma)'(0)} \\ &= ((\sigma \cdot \sigma)'(0) + c_1 y) \left( \frac{1}{(\sigma \cdot \sigma)'(0)} + \frac{c_2 y}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \frac{c_2^2 y^2}{4} \right) \left( \frac{4\mu(0) - (\sigma \cdot \sigma)'(0) + 4c_1 y}{2(\sigma \cdot \sigma)'(0) - c_1 y} \right) - \frac{4\mu(0) - (\sigma \cdot \sigma)'(0)}{2(\sigma \cdot \sigma)'(0)}. \end{aligned} \quad (171)$$

Analogously, we obtain from (162) that for all  $y \in (0, \varepsilon)$  it holds that

$$\begin{aligned} \phi(y) &= \int_0^y \frac{1}{\sigma(z)} dz = \int_0^y \frac{1}{\sqrt{\int_0^z (\sigma \cdot \sigma)'(x) dx}} dz \geq \int_0^y \frac{1}{\sqrt{(\sigma \cdot \sigma)'(0)z + c_1 z^2}} dz \\ &= \int_0^y \frac{1}{\sqrt{(\sigma \cdot \sigma)'(0)z}} dz + \int_0^y \frac{\sqrt{(\sigma \cdot \sigma)'(0)z} - \sqrt{(\sigma \cdot \sigma)'(0)z + c_1 z^2}}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{(\sigma \cdot \sigma)'(0)z + c_1 z^2}} dz \\ &= \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - \int_0^y \frac{c_1 z^2}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{(\sigma \cdot \sigma)'(0)z + c_1 z^2} (\sqrt{(\sigma \cdot \sigma)'(0)z} + \sqrt{(\sigma \cdot \sigma)'(0)z + c_1 z^2})} dz \\ &\geq \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - \int_0^y \frac{c_1 z^2}{\sqrt{(\sigma \cdot \sigma)'(0)z} \sqrt{\frac{1}{4}(\sigma \cdot \sigma)'(0)z} (\sqrt{(\sigma \cdot \sigma)'(0)z} + \sqrt{\frac{1}{4}(\sigma \cdot \sigma)'(0)z})} dz \\ &= \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - \int_0^y \frac{4c_1 z^2}{3((\sigma \cdot \sigma)'(0)z)^{3/2}} dz = \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - \frac{8c_1 y^{3/2}}{9((\sigma \cdot \sigma)'(0))^{3/2}} = \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - c_2 y^{3/2}. \end{aligned} \quad (172)$$

Next note that (162) shows that for all  $y \in (0, \varepsilon)$  it holds that  $\frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - c_2 y^{3/2} > 0$ . This, (162) and (163) imply that for all  $y \in (0, \varepsilon)$  it holds that

$$\begin{aligned} &(\phi(y))^2 \cdot \left( \frac{\left( \mu - \frac{(\sigma \cdot \sigma)'}{4} \right) (\sigma \cdot \sigma)'}{2(\sigma \cdot \sigma)} \right)(y) - \alpha \geq \left( \frac{2\sqrt{y}}{\sqrt{(\sigma \cdot \sigma)'(0)}} - c_2 y^{3/2} \right)^2 \frac{\left( \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} - c_1 y \right) ((\sigma \cdot \sigma)'(0) - c_1 y)}{2 \int_0^y (\sigma \cdot \sigma)'(x) dx} - \alpha \\ &\geq \left( \frac{4y}{(\sigma \cdot \sigma)'(0)} - \frac{4c_2 y^2}{\sqrt{(\sigma \cdot \sigma)'(0)}} + c_2^2 y^3 \right) \frac{\left( \mu(0) - \frac{(\sigma \cdot \sigma)'(0)}{4} - c_1 y \right) ((\sigma \cdot \sigma)'(0) - c_1 y)}{2(\sigma \cdot \sigma)'(0)y + c_1 y^2} - \frac{4\mu(0) - (\sigma \cdot \sigma)'(0)}{2(\sigma \cdot \sigma)'(0)} \\ &= ((\sigma \cdot \sigma)'(0) - c_1 y) \left( \frac{1}{(\sigma \cdot \sigma)'(0)} - \frac{c_2 y}{\sqrt{(\sigma \cdot \sigma)'(0)}} + \frac{c_2^2 y^2}{4} \right) \left( \frac{4\mu(0) - (\sigma \cdot \sigma)'(0) - 4c_1 y}{2(\sigma \cdot \sigma)'(0) + c_1 y} \right) - \frac{4\mu(0) - (\sigma \cdot \sigma)'(0)}{2(\sigma \cdot \sigma)'(0)}. \end{aligned} \quad (173)$$

Next observe that (171) and (173) prove that

$$\limsup_{(0, \varepsilon) \ni x \rightarrow 0} \left[ \frac{1}{x} \left| (\phi(x))^2 \cdot \left( \frac{\left( \mu - \frac{(\sigma \cdot \sigma)'}{4} \right) (\sigma \cdot \sigma)'}{2(\sigma \cdot \sigma)} \right)(x) - \alpha \right| \right] < \infty. \quad (174)$$

Finally, we note that (169), the continuity of  $\phi$ , the fact that  $\phi(0) = 0$ , the assumptions that  $\mu \in C^1([0, \infty), \mathbb{R})$  and  $(\sigma \cdot \sigma) \in C^2([0, \infty), [0, \infty))$  and (174) and (166) ensure that

$$\limsup_{(0, \varepsilon) \ni x \rightarrow 0} \left| g'(x) + \frac{\alpha}{x^2} \right| = \limsup_{(0, \varepsilon) \ni x \rightarrow 0} \left| -g'(\phi(x)) - \frac{\alpha}{(\phi(x))^2} \right|$$

$$\begin{aligned}
&\leq \limsup_{(0,\varepsilon)\ni x\rightarrow 0} \left( |\mu'(x)| + \left| \frac{(\sigma\cdot\sigma)''(x)}{4} \right| \right) + \limsup_{(0,\varepsilon)\ni x\rightarrow 0} \left| \left( \frac{\left( \mu - \frac{(\sigma\cdot\sigma)'}{4} \right) (\sigma\cdot\sigma)'}{2(\sigma\cdot\sigma)} \right) (x) - \frac{\alpha}{(\phi(x))^2} \right| \\
&= |\mu'(0)| + \left| \frac{(\sigma\cdot\sigma)''(0)}{4} \right| + \left[ \limsup_{(0,\varepsilon)\ni x\rightarrow 0} \frac{1}{x} \left| (\phi(x))^2 \left( \frac{\left( \mu - \frac{(\sigma\cdot\sigma)'}{4} \right) (\sigma\cdot\sigma)'}{2(\sigma\cdot\sigma)} \right) (x) - \alpha \right| \right] \left[ \limsup_{(0,\varepsilon)\ni x\rightarrow 0} \left| \frac{x}{(\phi(x))^2} \right| \right] < \infty.
\end{aligned} \tag{175}$$

This finishes the proof of Lemma 3.11.  $\square$

**Lemma 3.12.** *Let  $\alpha \in (0, \infty)$ ,  $\mu \in C^1([0, \infty), \mathbb{R})$ ,  $\sigma \in C([0, \infty), [0, \infty))$  satisfy  $\sigma(0) = 0$ ,  $\sigma((0, \infty)) \subseteq (0, \infty)$ ,  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$ ,  $(\sigma \cdot \sigma)'(0) > 0$ ,  $\alpha = \frac{2\mu(0)}{(\sigma\cdot\sigma)'(0)} - \frac{1}{2}$ , let  $\phi: [0, \infty) \rightarrow [0, \infty)$  and  $g: (0, \infty) \rightarrow \mathbb{R}$  be functions defined by  $\phi(y) := \int_0^y \frac{1}{\sigma(z)} dz$  and  $g(z) := \left( \frac{\mu - 1/4(\sigma\cdot\sigma)'}{\sigma} \right) (\phi^{-1}(z))$  for all  $y \in [0, \infty)$ ,  $z \in (0, \infty)$ , let  $L := [\sup_{z \in (0, \infty)} g'(z)]^+ \in [0, \infty]$  and assume that*

$$\inf_{\rho \in [1, \infty)} \limsup_{x \rightarrow \infty} \left[ \frac{[\mu(x)]^+}{x} + \frac{\sigma(x)}{x} + \frac{|\mu(x)|}{x^\rho} + [g'(x)]^+ + \frac{|g'(x)|}{x^\rho} \right] < \infty. \tag{176}$$

Then  $\phi$  is well-defined and bijective,  $\phi^{-1}|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ ,  $g$  is well-defined and continuously differentiable,  $L < \infty$ , it holds for all  $t \in (0, 1/L)$  that the function  $(0, \infty) \ni x \mapsto x - tg(x) \in \mathbb{R}$  is bijective, it holds that

$$\lim_{(0, \infty) \ni x \rightarrow 0} \frac{|\mu(x) - \mu(0)| + |(\sigma\cdot\sigma)'(x) - (\sigma\cdot\sigma)'(0)|}{\sigma(x)} < \infty \tag{177}$$

and it holds that there exists a real number  $c \in [0, \infty)$  such that for all  $x, y \in (0, \infty)$  it holds that

$$(x - y)(g(x) - g(y)) \leq L(x - y)^2, \tag{178}$$

$$|g(x) - g(y)| \leq c|x - y|(1 + \frac{1}{xy} + x^c + y^c), \tag{179}$$

$$\alpha - c(x + x^c) \leq xg(x) \leq c(1 + x^2). \tag{180}$$

*Proof of Lemma 3.12.* The assumption that  $\limsup_{x \rightarrow \infty} \frac{\sigma(x)}{x} < \infty$  implies that  $\int_1^\infty \frac{1}{\sigma(z)} dz = \infty$ . Moreover, the assumptions that  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$  and that  $\sigma((0, \infty)) \subseteq (0, \infty)$  show that  $\sigma|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . Lemma 3.11 implies that  $\phi$  is well-defined and bijective, that  $g \in C^1((0, \infty), \mathbb{R})$ , that

$$\lim_{(0, \infty) \ni x \rightarrow 0} xg(x) = \alpha \quad \text{and that} \quad \limsup_{(0, \infty) \ni x \rightarrow 0} \left| g'(x) + \frac{\alpha}{x^2} \right| < \infty. \tag{181}$$

This implies that  $\liminf_{(0, \infty) \ni x \rightarrow 0} g(x) = \infty$  and that  $\limsup_{(0, \infty) \ni x \rightarrow 0} g'(x) < \infty$ . Moreover, Lemma 3.11 yields that  $\lim_{(0, \infty) \ni x \rightarrow 0} \frac{\sigma(x)}{\sqrt{x}} = \sqrt{(\sigma \cdot \sigma)'(0)} \in (0, \infty)$ . Hence, there exist real numbers  $\varepsilon \in (0, 1)$ ,  $\gamma \in (0, \infty)$  such that for all  $x \in (0, \varepsilon)$  it holds that  $\sigma(x) \geq \gamma\sqrt{x}$ . This,  $\mu \in C^1([0, \infty), \mathbb{R})$  and  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$  imply that there exist real numbers  $\varepsilon \in (0, 1)$ ,  $\gamma \in (0, \infty)$  such that

$$\begin{aligned}
\lim_{(0, \infty) \ni x \rightarrow 0} \frac{|\mu(x) - \mu(0)| + |(\sigma\cdot\sigma)'(x) - (\sigma\cdot\sigma)'(0)|}{\sigma(x)} &\leq \lim_{(0, \varepsilon) \ni x \rightarrow 0} \frac{|\mu(x) - \mu(0)| + |(\sigma\cdot\sigma)'(x) - (\sigma\cdot\sigma)'(0)|}{\gamma x} \\
&= \frac{|\mu'(0)| + |(\sigma\cdot\sigma)''(0)|}{\gamma} < \infty.
\end{aligned} \tag{182}$$

Observe that  $\phi$  is strictly increasing and continuous and that  $\phi|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . This, the fact that  $\phi(0) = 0$ , the continuity of  $\phi$  and the fact that  $\int_1^\infty \frac{1}{\sigma(z)} dz = \infty$  imply that  $\phi$  is bijective, that  $\phi^{-1}: [0, \infty) \rightarrow [0, \infty)$  is strictly increasing and continuous and that  $\phi^{-1}|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ . Next note that the assumption that  $\limsup_{(0, \infty) \ni x \rightarrow 0} g'(x) < \infty$ , (176) and the fact that  $g \in C^1((0, \infty), \mathbb{R})$  yield that  $L < \infty$ . Moreover, observe that for all  $x, y \in (0, \infty)$  it holds that

$$(x - y)(g(x) - g(y)) \leq L(x - y)^2. \tag{183}$$

In the next step we note that (181) and (176) imply that there exists  $\lambda \in [3, \infty)$ , which we fix for the rest of this proof, such that for all  $x \in (0, \infty)$ ,  $y \in [1, \infty)$ ,  $z \in (0, 1)$  it holds that

$$\left| g'(x) + \frac{\alpha}{x^2} \right| \leq \lambda(1 + x^\lambda), \quad |g'(y)| \leq \lambda y^{(\lambda-2)}, \quad zg(z) \leq \lambda \quad \text{and} \quad \left| g'(z) + \frac{\alpha}{z^2} \right| \leq \lambda. \tag{184}$$

Moreover, the mean value theorem implies that for every  $x, y \in (0, \infty)$  with  $x < y$  there exists a real number  $\xi \in (x, y)$  such that  $\frac{g(x) - \frac{\alpha}{x} - g(y) + \frac{\alpha}{y}}{x - y} = g'(\xi) + \frac{\alpha}{\xi^2}$ . This and (184) show that for all  $x, y \in (0, \infty)$  it holds that

$$\begin{aligned}
|g(x) - g(y)| &\leq \left| g(x) - \frac{\alpha}{x} - g(y) + \frac{\alpha}{y} \right| + \alpha \left| \frac{1}{x} - \frac{1}{y} \right| \leq \lambda|x - y|(1 + x^\lambda + y^\lambda) + \alpha|x - y| \frac{1}{xy} \\
&\leq (\alpha + \lambda)|x - y|(1 + \frac{1}{xy} + x^\lambda + y^\lambda).
\end{aligned} \tag{185}$$

In the next step we note that (184) ensures that for all  $x \in [1, \infty)$  it holds that

$$xg(x) = xg(1) + x \int_1^x g'(z) dz \geq xg(1) - \lambda x \int_1^x z^{(\lambda-2)} dz = xg(1) - \lambda x \frac{(x^{(\lambda-1)} - 1)}{(\lambda-1)} \geq \alpha - (|g(1)| + \alpha + \lambda)x^\lambda \tag{186}$$

and that for all  $x \in (0, 1)$  it holds that

$$xg(x) = xg(1) + x \int_1^x g'(z) dz + \frac{\alpha}{x^2} dz - x \int_1^x \frac{\alpha}{z^2} dz \geq x(g(1) - \alpha) + \alpha - x \int_1^x \lambda dz \geq \alpha - (|g(1)| + \lambda + \alpha)x. \quad (187)$$

Moreover, note that the fact that  $L \in [0, \infty)$  shows that for all  $x \in [1, \infty)$  it holds that

$$xg(x) = xg(1) + x \int_1^x g'(z) dz \leq xg(1) + Lx(x-1) \leq (L + |g(1)|)(1+x^2). \quad (188)$$

In the next step we define a real number  $c \in \mathbb{R}$  through  $c := \alpha + L + |g(1)| + \lambda$  and we observe that (184), (186), (187) and (188) show that for all  $x \in (0, \infty)$  it holds that

$$\alpha - c(x + x^c) \leq xg(x) \leq c(1 + x^2). \quad (189)$$

Observe that  $g \in C^1((0, \infty), \mathbb{R})$  implies that for all  $x \in (1, \infty)$ ,  $t \in [0, \infty)$  it holds that

$$\begin{aligned} x - tg(x) &= x - t[g(x) - g(1) + g(1)] \geq x - t[L(x-1) + g(1)] \\ &= x(1 - tL) - t(g(1) - L). \end{aligned} \quad (190)$$

This implies that for all  $t \in (0, \frac{1}{L})$  it holds that  $\lim_{x \rightarrow \infty} (x - tg(x)) = \infty$ . Combining this and the fact that  $\liminf_{x \searrow 0} g(x) = \infty$  with the continuity of the function  $g$  yields that for all  $t \in (0, \frac{1}{L})$  it holds that the function  $(0, \infty) \ni x \mapsto x - tg(x) \in \mathbb{R}$  is surjective. This together with Lemma 3.4 ensures that for all  $t \in (0, \frac{1}{L})$  it holds that the function  $(0, \infty) \ni x \mapsto x - tg(x) \in \mathbb{R}$  is bijective. The proof of Lemma 3.12 is thus completed.  $\square$

**Theorem 3.13.** *Let  $T, \alpha \in (0, \infty)$ ,  $\mu \in C^1([0, \infty), \mathbb{R})$ ,  $\sigma \in C([0, \infty), [0, \infty))$  satisfy  $\sigma(0) = 0$ ,  $\sigma((0, \infty)) \subseteq (0, \infty)$ ,  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$ ,  $(\sigma \cdot \sigma)'(0) > 0$ ,  $\alpha = \frac{2\mu(0)}{(\sigma \cdot \sigma)'(0)} - \frac{1}{2}$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying*

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s \quad (191)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , let  $\phi: [0, \infty) \rightarrow [0, \infty)$  and  $g: [0, \infty) \rightarrow \mathbb{R}$  be functions defined by  $\phi(y) := \int_0^y \frac{1}{\sigma(z)} dz$ ,  $g(0) := 0$  and  $g(z) := \left( \frac{\mu^{-1/4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(z))$  for all  $y \in [0, \infty)$ ,  $z \in (0, \infty)$ , let  $L := \left[ \sup_{z \in (0, \infty)} g'(z) \right]^+ \in [0, \infty]$  and assume that  $\sup_{t \in [0, T]} \mathbb{E}[\phi(X_0)^r + |\phi(X_t)|^{-q}] < \infty$  for all  $r \in \mathbb{R}$ ,  $q \in [1, 1 + 2\alpha)$  and

$$\inf_{\rho \in [1, \infty)} \limsup_{x \rightarrow \infty} \left[ \frac{[\mu(x)]^+}{x} + \frac{\sigma(x)}{x} + \frac{|\mu(x)|}{x^\rho} + \frac{\sigma(x)}{1 + (\phi(x))^\rho} + [g'(x)]^+ + \frac{|g'(x)|}{x^\rho} \right] < \infty. \quad (192)$$

Then  $L < \infty$  and there exists a unique family  $Y^h: (\mathbb{N}_0 \cap [0, T/h]) \times \Omega \rightarrow [0, \infty)$ ,  $h \in (0, T] \cap (0, 1/L)$ , of stochastic processes satisfying  $Y_0^h = \phi(X_0)$  and

$$Y_n^h = Y_{n-1}^h + g(Y_n^h)h + W_{nh} - W_{(n-1)h} \quad (193)$$

for all  $n \in \mathbb{N} \cap [0, T/h]$ ,  $h \in (0, T] \cap (0, 1/L)$  and it holds for all  $\varepsilon \in (0, \infty)$ ,  $p \in [1, \infty)$  that

$$\sup_{h \in (0, T] \cap [0, 1/(4L)]} \left\| h^{\left( \varepsilon - \frac{(\alpha \wedge 1/2)}{p} \right)} \right\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \|X_{nh} - \phi^{-1}(Y_n^h)\|_{L^p(\Omega; \mathbb{R})} < \infty. \quad (194)$$

*Proof of Theorem 3.13.* It follows from Lemma 3.12 that  $\phi$  is well-defined and bijective, that  $\phi^{-1}|_{(0, \infty)} \in C^1((0, \infty), [0, \infty))$ , that  $g$  is well-defined, that  $g|_{(0, \infty)}$  is continuously differentiable, that  $L < \infty$  and that

$$\lim_{(0, \infty) \ni x \rightarrow 0} \frac{|\mu(x) - \mu(0)| + |(\sigma \cdot \sigma)'(x) - (\sigma \cdot \sigma)'(0)|}{\sigma(x)} < \infty. \quad (195)$$

Furthermore, Lemma 3.12 implies that there exists a real number  $c \in \mathbb{R}$ , which we fix for the rest of this proof, such that for all  $x, y \in (0, \infty)$  it holds that

$$(x - y)(g(x) - g(y)) \leq L(x - y)^2, \quad (196)$$

$$|g(x) - g(y)| \leq c|x - y|(1 + \frac{1}{xy} + x^c + y^c), \quad (197)$$

$$\alpha - c(x + x^c) \leq xg(x) \leq c(1 + x^2). \quad (198)$$

Moreover, Lemma 3.12 shows that for all  $t \in (0, 1/L)$  it holds that the function  $(0, \infty) \ni x \mapsto x - tg(x) \in \mathbb{R}$  is bijective. This proves that there exists a unique family  $Y^h: (\mathbb{N}_0 \cap [0, T/h]) \times \Omega \rightarrow [0, \infty)$ ,  $h \in (0, T] \cap (0, 1/L)$ , of stochastic processes satisfying  $Y_0^h = \phi(X_0)$  and

$$Y_n^h = Y_{n-1}^h + g(Y_n^h)h + W_{nh} - W_{(n-1)h} \quad (199)$$

for all  $n \in \mathbb{N} \cap [0, T/h]$ ,  $h \in (0, T] \cap (0, 1/L)$ . In the next step, we define a mapping  $Z: [0, T] \times \Omega \rightarrow [0, \infty)$  by  $Z_t := \phi(X_t)$  for all  $t \in [0, T]$ . Note that  $Z$  is an adapted stochastic process with continuous sample paths.

Moreover, observe that the assumption that  $\sup_{t \in [0, T]} \mathbb{E}[(Z_t)^{-1}] < \infty$  implies that  $\sup_{t \in [0, T]} \mathbb{P}[Z_t = 0] = 0$ . Next note that Lemma 3.2, (195),  $\sigma(0) = 0$  and  $\mu(0) - \frac{1}{4}(\sigma \cdot \sigma)'(0) > 0$  yield that

$$\int_0^T \left| \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(Z_s)) \mathbb{1}_{\{Z_s \in (0, \infty)\}} \right| ds < \infty \quad (200)$$

$\mathbb{P}$ -a.s. and

$$Z_t = Z_0 + \int_0^t \left( \frac{\mu - \frac{1}{4}(\sigma \cdot \sigma)'}{\sigma} \right) (\phi^{-1}(Z_s)) \mathbb{1}_{\{Z_s \in \phi((0, \infty))\}} ds + W_t = Z_0 + \int_0^t g(Z_s) ds + W_t \quad (201)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . Lemma 2.6 together with (198) and  $\mathbb{E}[(Z_0)^q] < \infty$  for all  $q \in (0, \infty)$  implies that for all  $q \in (0, \infty)$  it holds that  $\sup_{t \in [0, T]} \mathbb{E}[(Z_t)^q] < \infty$ . Then Lemma 2.7 yields that for all  $q \in (0, \infty)$  it holds that

$$\left\| \sup_{s \in [0, T]} Z_s \right\|_{L^q(\Omega; \mathbb{R})} < \infty. \quad (202)$$

In the next step we note that (198) and the assumption that  $\mathbb{E}[(Z_0)^q] < \infty$  for all  $q \in \mathbb{R}$  show that for all  $q \in (0, \infty)$  it holds that  $\mathbb{E}[|g(Z_0)|^q] < \infty$ . Corollary 3.8 (applied with  $O = (0, \infty)$  and  $\xi = Z_0 + \mathbb{1}_{\{Z_0=0\}}$ ),  $\mathbb{P}[Z_0 = 0] = 0$ , (196) and (198) prove that for all  $q \in (0, \infty)$  it holds that

$$\sup_{h \in (0, T] \cap (0, \frac{1}{4L}]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} Y_n^h \right\|_{L^q(\Omega; \mathbb{R})} < \infty. \quad (203)$$

Corollary 3.9 together with (196) and (197) shows that for all  $\varepsilon, \kappa \in (0, \infty)$ ,  $\gamma \in (0, 1]$  it holds that

$$\begin{aligned} & \sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ h^{-\gamma(\frac{1}{2} - \frac{\varepsilon}{2\gamma})} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Z_{nh} - Y_n^h| \right\|_{L^1(\Omega; \mathbb{R})} \right] \\ & \leq 2Tc \sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ \left( \frac{1}{1-hL} \right)^{\lceil T/h \rceil} h^{-\gamma(\frac{1}{2} - \frac{\varepsilon}{2\gamma})} \left\| \sup_{u \in [0, T]} |Z_u - Z_{\lceil u \rceil(0, h, 2h, \dots, \lceil T/h \rceil h)}| \right\|_{L^{\gamma(1+1/\kappa)}(\Omega; \mathbb{R})}^\gamma \right] \\ & \cdot \sup_{u \in [0, T]} \left[ \left\| \frac{2}{(Z_u)^{1+\gamma}} \right\|_{L^{1+\kappa}(\Omega; \mathbb{R})} + \frac{\gamma}{1+\gamma} \left\| (Z_u)^{\frac{1}{\gamma}-\gamma} \right\|_{L^{1+\kappa}(\Omega; \mathbb{R})} + 2 \left\| (Z_u)^{c+1-\gamma} \right\|_{L^{1+\kappa}(\Omega; \mathbb{R})} + \frac{1}{1+\gamma} \right]. \end{aligned} \quad (204)$$

Estimate (204) (applied with  $\varepsilon \in (0, 2\alpha \wedge 1)$ ,  $\gamma = 2\alpha \wedge 1 - \varepsilon$ ,  $\kappa = \frac{1}{2}(\frac{1+2\alpha}{1+\gamma} - 1)$ ), Theorem 2.13, the assumption that  $\mu \in C^1([0, \infty), \mathbb{R})$ , the assumption that  $\sigma \cdot \sigma \in C^2([0, \infty), [0, \infty))$ , (192), (202) and the assumption that  $\sup_{t \in [0, T]} \mathbb{E}[(Z_t)^{-q}] < \infty$  for all  $q \in [1, 1+2\alpha]$  show that for all  $\varepsilon \in (0, 2\alpha \wedge 1)$  it holds that

$$\sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ h^{(\varepsilon - \min\{\alpha, 1/2\})} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Z_{nh} - Y_n^h| \right\|_{L^1(\Omega; \mathbb{R})} \right] < \infty. \quad (205)$$

Remark 3.10, (205), (202) and (203) yield that for all  $p \in [1, \infty)$ ,  $\varepsilon \in (0, \min\{2\alpha, 1\})$ ,  $\kappa \in (0, \infty)$  it holds that

$$\begin{aligned} & \sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ h^{-\lceil \frac{\min\{\alpha, 1/2\} - \varepsilon}{p(1+\kappa)} \rceil} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Z_{nh} - Y_n^h| \right\|_{L^p(\Omega; \mathbb{R})} \right] \\ & \leq \sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left( \left[ h^{(\varepsilon - \min\{\alpha, 1/2\})} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Z_{nh} - Y_n^h| \right\|_{L^1(\Omega; \mathbb{R})} \right]^{\frac{1}{p(1+\kappa)}} \right. \\ & \cdot \left. \left[ \left\| \sup_{s \in [0, T]} |Z_s| \right\|_{L^{p+\frac{p-1}{\kappa}}(\Omega; \mathbb{R})}^{1-\frac{1}{p(1+\kappa)}} + \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Y_n^h| \right\|_{L^{p+\frac{p-1}{\kappa}}(\Omega; \mathbb{R})}^{1-\frac{1}{p(1+\kappa)}} \right] \right) < \infty. \end{aligned} \quad (206)$$

Next note that estimate (192) proves that there exists a real number  $\rho \in \mathbb{R}$ , which we fix for the rest of this proof, such that for all  $x \in [0, \infty)$  it holds that  $\sigma(x) \leq \rho(1 + (\phi(x))^\rho)$ . This together with the monotonicity and the continuity of  $\phi$  shows that for all  $p \in [1, \infty)$ ,  $h \in (0, T] \cap (0, \frac{1}{4L}]$ ,  $\delta \in (0, \infty)$  it holds that

$$\begin{aligned} & \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \sup_{z \in [X_{nh} \wedge \phi^{-1}(Y_n^h), X_{nh} \vee \phi^{-1}(Y_n^h)]} \sigma(z) \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} \\ & \leq \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \sup_{z \in [X_{nh} \wedge \phi^{-1}(Y_n^h), X_{nh} \vee \phi^{-1}(Y_n^h)]} \rho(1 + (\phi(z))^\rho) \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} \\ & \leq \rho \left[ \left\| \sup_{s \in [0, T]} [1 + (Z_s)^\rho] \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} + \sup_{\tilde{h} \in (0, T] \cap [0, \frac{1}{4L}]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/\tilde{h}]} [1 + (Y_n^{\tilde{h}})^\rho] \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} \right]. \end{aligned} \quad (207)$$

Hölder's inequality and (207) show that for all  $p \in [1, \infty)$ ,  $h \in (0, T] \cap (0, \frac{1}{4L}]$ ,  $\delta \in (0, \infty)$  it holds that

$$\left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |X_{nh} - \phi^{-1}(Y_n^h)| \right\|_{L^p(\Omega; \mathbb{R})} = \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \left| \int_{X_{nh}}^{\phi^{-1}(Y_n^h)} \frac{\sigma(z)}{\sigma(z)} dz \right| \right\|_{L^p(\Omega; \mathbb{R})}$$

$$\begin{aligned}
&\leq \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \sup_{z \in [X_{nh} \wedge \phi^{-1}(Y_n^h), X_{nh} \vee \phi^{-1}(Y_n^h)]} \sigma(z) \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} \left| \int_{X_{nh}}^{\phi^{-1}(Y_n^h)} \frac{1}{\sigma(z)} dz \right| \right\|_{L^{p(1+\delta)}(\Omega; \mathbb{R})} \\
&\leq \rho \left[ \left\| \sup_{s \in [0, T]} [1 + (Z_s)^\rho] \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} + \sup_{\tilde{h} \in (0, T] \cap [0, \frac{1}{4L}]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/\tilde{h}]} [1 + (Y_n^{\tilde{h}})^\rho] \right\|_{L^{p(1+1/\delta)}(\Omega; \mathbb{R})} \right] \\
&\quad \cdot \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |Z_{nh} - Y_n^h| \right\|_{L^{p(1+\delta)}(\Omega; \mathbb{R})}. \tag{208}
\end{aligned}$$

This, (202), (203) and (206) imply that for all  $p \in [1, \infty)$ ,  $\varepsilon \in (0, \min\{2\alpha, 1\})$ ,  $\kappa, \delta \in (0, \infty)$  it holds that

$$\sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ h^{-\left[\frac{\min\{\alpha, 1/2\} - \varepsilon}{p(1+\delta)(1+\kappa)}\right]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |X_{nh} - \phi^{-1}(Y_n^h)| \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \tag{209}$$

This proves that for all  $p \in [1, \infty)$ ,  $\varepsilon \in (0, \frac{\min\{\alpha, 1/2\}}{p})$  it holds that

$$\sup_{h \in (0, T] \cap [0, \frac{1}{4L}]} \left[ h^{-\left[\frac{\alpha \wedge 1/2}{p} - \varepsilon\right]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |X_{nh} - \phi^{-1}(Y_n^h)| \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \tag{210}$$

The proof of Theorem 3.13 is thus completed.  $\square$

Theorem 3.13 uses the assumption that suitable inverse moments of a transformation of the solution process of the SDE (191) is finite, that is, in the setting of Theorem 3.13 that the quantity  $\sup_{t \in [0, T]} \mathbb{E}[|\phi(X_t)|^{-q}]$  is finite for all  $q \in [1, 1 + 2\alpha)$ . The next result, Lemma 3.14, gives a sufficient condition to ensure finiteness of suitable inverse moments of Cox-Ingersoll-Ross processes. Lemma 3.14 extends and is based on Lemma A.1 in Bossy & Diop [6].

**Lemma 3.14.** *Let  $T, \beta, \delta \in (0, \infty)$ ,  $\gamma \in \mathbb{R}$ ,  $p \in (0, 2\delta/\beta^2)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying*

$$X_t = X_0 + \int_0^t \delta - \gamma X_s ds + \int_0^t \beta \sqrt{X_s} dW_s \tag{211}$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$  and assume that  $\inf_{\varepsilon \in (0, 2\delta/\beta^2 - p)} \sup_{u \in [1, \infty)} u^{(p+\varepsilon)} \mathbb{E}[\exp(-uX_0)] < \infty$ . Then it holds that  $\sup_{t \in [0, T]} \mathbb{E}[(X_t)^{-p}] < \infty$ .

*Proof of Lemma 3.14.* First of all, note that the assumption  $\inf_{\varepsilon \in (0, 2\delta/\beta^2 - p)} \sup_{u \in [1, \infty)} u^{(p+\varepsilon)} \mathbb{E}[\exp(-uX_0)] < \infty$  implies that  $\inf_{\varepsilon \in (0, 2\delta/\beta^2 - p)} \sup_{u \in (0, \infty)} u^{(p+\varepsilon)} \mathbb{E}[\exp(-uX_0)] < \infty$ . This shows that there exist real numbers  $C \in (0, \infty)$ ,  $\varepsilon \in (0, 2\delta/\beta^2 - p)$ , which we fix for the rest of this proof, such that for all  $u \in (0, \infty)$  it holds that  $\mathbb{E}[\exp(-uX_0)] \leq C u^{-(p+\varepsilon)}$ . This shows that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
&\int_1^\infty u^{(p-1)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{-\frac{2\delta}{\beta^2}} \mathbb{E} \left[ \exp \left( -\frac{e^{-\gamma t}}{\frac{u\beta^2}{2\gamma}(1 - e^{-\gamma t}) + 1} X_0 u \right) \right] du \\
&\leq C \int_1^\infty u^{(p-1)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{-\frac{2\delta}{\beta^2}} \left[ \frac{e^{-\gamma t}}{\frac{u\beta^2}{2\gamma}(1 - e^{-\gamma t}) + 1} u \right]^{-(p+\varepsilon)} du \\
&= C e^{\gamma t(p+\varepsilon)} \int_1^\infty u^{-(1+\varepsilon)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{\left[p+\varepsilon - \frac{2\delta}{\beta^2}\right]} du \leq C e^{\gamma t(p+\varepsilon)} \int_1^\infty u^{-(1+\varepsilon)} du = \frac{C}{\varepsilon} e^{\gamma t(p+\varepsilon)}.
\end{aligned} \tag{212}$$

This, line 8 in the proof of Lemma A.1 in Bossy & Diop [6] and Fubini's theorem imply that for all  $t \in [0, T]$  it holds that

$$\begin{aligned}
\mathbb{E}[(X_t)^{-p}] &= \mathbb{E} \left[ \frac{1}{\Gamma(p)} \int_0^\infty u^{(p-1)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{-\frac{2\delta}{\beta^2}} \exp \left( -\frac{e^{-\gamma t}}{\frac{u\beta^2}{2\gamma}(1 - e^{-\gamma t}) + 1} X_0 u \right) du \right] \\
&= \frac{1}{\Gamma(p)} \left( \int_0^1 u^{(p-1)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{-\frac{2\delta}{\beta^2}} \mathbb{E} \left[ \exp \left( -\frac{e^{-\gamma t}}{\frac{u\beta^2}{2\gamma}(1 - e^{-\gamma t}) + 1} X_0 u \right) \right] du \right. \\
&\quad \left. + \int_1^\infty u^{(p-1)} \left[ \frac{u\beta^2}{2\gamma} (1 - e^{-\gamma t}) + 1 \right]^{-\frac{2\delta}{\beta^2}} \mathbb{E} \left[ \exp \left( -\frac{e^{-\gamma t}}{\frac{u\beta^2}{2\gamma}(1 - e^{-\gamma t}) + 1} X_0 u \right) \right] du \right) \\
&\leq \frac{1}{\Gamma(p)} \left( \int_0^1 u^{(p-1)} du + \frac{C}{\varepsilon} e^{\gamma t(p+\varepsilon)} \right) = \frac{1}{\Gamma(p)} \left( \frac{1}{p} + \frac{C}{\varepsilon} e^{\gamma t(p+\varepsilon)} \right) < \infty.
\end{aligned} \tag{213}$$

This finishes the proof of Lemma 3.14.  $\square$

**Corollary 3.15.** *Let  $T, \beta \in (0, \infty)$ ,  $\gamma \in \mathbb{R}$ ,  $\delta \in (\beta^2/4, \infty)$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$  be a stochastic basis, let  $W: [0, T] \times \Omega \rightarrow \mathbb{R}$  be a standard  $(\mathcal{F}_t)_{t \in [0, T]}$ -Brownian motion, let  $X: [0, T] \times \Omega \rightarrow [0, \infty)$  be an adapted stochastic process with continuous sample paths satisfying  $\inf_{r \in (0, 2\delta/\beta^2 - q)} \sup_{u \in [1, \infty)} u^{(q+r)} \mathbb{E}[\exp(-uX_0)] + \mathbb{E}[(X_0)^s] < \infty$  for all  $q \in [1/2, 2\delta/\beta^2)$ ,  $s \in \mathbb{R}$  and*

$$X_t = X_0 + \int_0^t \delta - \gamma X_s ds + \int_0^t \beta \sqrt{X_s} dW_s \quad (214)$$

$\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . Then there exists a unique family  $Y^h: [0, \lfloor T/h \rfloor h] \times \Omega \rightarrow [0, \infty)$ ,  $h \in (0, T] \cap (0, 2/(-\gamma)^+)$ , of mappings satisfying for all  $h \in (0, T] \cap (0, 2/(-\gamma)^+)$ ,  $n \in \mathbb{N}_0 \cap [0, T/h - 1]$ ,  $t \in (nh, (n+1)h]$  that  $Y_0^h = X_0$  and

$$Y_t^h = \left[ n + 1 - \frac{t}{h} \right] Y_{nh}^h + \left[ \frac{t}{h} - n \right] \left[ \frac{(Y_{nh}^h)^{1/2 + \frac{\beta}{2}} (W_{nh+h} - W_{nh}) + \sqrt{[(Y_{nh}^h)^{1/2 + \frac{\beta}{2}} (W_{nh+h} - W_{nh})]^2 + (2+\gamma h)(\delta - \frac{\beta^2}{4})h}}{(2+\gamma h)} \right]^2 \quad (215)$$

and it holds for all  $p, \varepsilon \in (0, \infty)$  that

$$\sup_{h \in (0, T] \cap (0, 1/(-2\gamma)^+)} \left[ h^{\left[ \varepsilon - \frac{(2\delta/\beta^2) \wedge 1 - 1/2}{(p \vee 1)} \right]} \left\| \sup_{t \in [0, \lfloor T/h \rfloor h]} |X_t - Y_t^h| \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \quad (216)$$

*Proof of Corollary 3.15.* We fix  $p, \varepsilon \in (0, \infty)$  throughout this proof and we define a function  $\phi: [0, \infty) \rightarrow [0, \infty)$  by

$$\phi(x) := \int_0^x \frac{1}{\beta \sqrt{z}} dz = \frac{2}{\beta} \sqrt{x} \quad (217)$$

for all  $x \in [0, \infty)$ . Note that  $\phi$  is bijective and observe that for all  $x \in [0, \infty)$  it holds that  $\phi^{-1}(x) = \frac{\beta^2}{4} x^2$ . We define  $\alpha := \frac{2\delta}{\beta^2} - \frac{1}{2} \in (0, \infty)$  and we define a function  $g: [0, \infty) \rightarrow \mathbb{R}$  by  $g(0) := 0$  and  $g(x) := \frac{\alpha}{x} - \frac{\gamma x}{2}$  for all  $x \in (0, \infty)$ . The fact that  $g|_{(0, \infty)} \in C^1((0, \infty), \mathbb{R})$  and that for all  $x \in (0, \infty)$  it holds that  $g'(x) = -\frac{\alpha}{x^2} - \frac{\gamma}{2}$  proves that  $L := [\sup_{x \in (0, \infty)} g'(x)]^+ = (-\gamma/2)^+$  and that  $\limsup_{x \rightarrow \infty} |g'(x)|/x = 0$ . Lemma 3.14 implies that for all  $q \in [\frac{1}{2}, \frac{2\delta}{\beta^2})$  it holds that  $\sup_{t \in [0, T]} \mathbb{E}[(X_t)^{-q}] < \infty$ . This shows that for all  $q \in [1, 1 + 2\alpha)$  it holds that  $\sup_{t \in [0, T]} \mathbb{E}[\phi(X_t)^{-q}] < \infty$ . Now we apply Theorem 3.13 to obtain that there exists a unique family  $Y^h: \{0, h, 2h, \dots, \lfloor T/h \rfloor h\} \times \Omega \rightarrow [0, \infty)$ ,  $h \in (0, T] \cap (0, 1/L)$ , of mappings satisfying  $Y_0^h = X_0$  and

$$\phi(Y_{nh}^h) = \phi(Y_{(n-1)h}^h) + g(\phi(Y_{nh}^h)) h + W_{nh} - W_{(n-1)h} \quad (218)$$

for all  $n \in \mathbb{N} \cap [0, T/h]$ ,  $h \in (0, T] \cap (0, 1/L)$  and it holds for all  $\varepsilon, p \in (0, \infty)$  that

$$\sup_{h \in (0, T] \cap (0, 1/(-2\gamma)^+)} \left[ h^{\left[ \varepsilon - \frac{(2\delta/\beta^2) \wedge 1 - 1/2}{(p \vee 1)} \right]} \left\| \sup_{n \in \mathbb{N}_0 \cap [0, T/h]} |X_{nh} - Y_{nh}^h| \right\|_{L^p(\Omega; \mathbb{R})} \right] < \infty. \quad (219)$$

The solution of the implicit equation (218) is well-known (see (4) in Alfonsi [1]) and the linear interpolations of the resulting discrete-time processes satisfy equation (215). Finally, (219) together with the fact that for all  $p, \varepsilon \in (0, \infty)$  it holds that

$$\left\| \sup_{s, t \in [0, T], s \neq t} \left[ \frac{|X_t - X_s|}{|t - s|^{(\varepsilon - \frac{1}{2})}} \right] \right\|_{L^p(\Omega; \mathbb{R})} < \infty \quad (220)$$

implies (216). This finishes the proof of Corollary 3.15.  $\square$

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